Information Cascades and Threshold Implementation*

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Abstract

Economic activities such as crowdfunding often involve sequential interactions, observational learning, and project implementation contingent on achieving certain thresholds of support. We incorporate endogenous all-or-nothing (AoN) thresholds in a classical model of information cascade. Relative to standard settings, we find that the AoN feature effectively delegates early supporters' downside protection to a later "gate-keeper", leads to uni-directional cascades and prevents agents' herding on rejections. Consequently, information aggregation improves, and issuance becomes less under-priced, even when agents have the options to wait. More generally, endogenous AoN thresholds improve the financing efficiency of costly projects and the harnessing of decentralized information, and approaches the first-best information production and project selection as the crowd grows large.

JEL Classification: D81, D83, G12, G14, L26

Keywords: Information Cascade, Crowd-funding, All-or-nothing, Entrepreneurial Finance, Capital Markets, Information Aggregation.

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1 Introduction

Numerous economic and financial activities involve sequential actions from dispersed agents, and are prone to information cascades (Banerjee (1992); Bikhchandani, Hirshleifer, and Welch (1992)). Once a cascade starts, all subsequent individuals act independent of their private signals and information aggregation stops. Classical models tend to focus on the case of pure informational externalities by assuming that one agent's payoff is independent of the action of others. They leave out an important feature commonly observed in real-life: many projects or proposals are only implemented with a sufficient level of support — an "all-or-nothing" (AoN) threshold. AoN threshold is predominent on crowdfunding platforms and in venture financing; super-majority rule or q-rule is a common practice in many voting procedures; assurance contract or crowdaction in public goods provision is also characterized by sequential decisions and implementation thresholds (e.g., Bagnoli and Lipman (1989)); charitable projects need a minimum level of funding-raised to proceed. Other examples abound.

While a recent literature on sequential voting (e.g., Dekel and Piccione (2000) and Ali and Kartik (2006)) sheds light on how herding and the interdependence of voter payoffs interact, several desirable features are still under-explored. First, unlike the case of voting whereby an elected candidate or bill passed affect all agents regardless whether they voted in favor or against, in many situations such as crowdfunding, venture investment, and campaign contributions intended to buy favors, the implementation of the project only affects agents taking a particular action. Second, AoN thresholds in extant models are typically taken as exogenous, yet entrepreneurs or or campaign leaders frequently set contribution amounts and target thresholds for implementation. Third, the Internet and latest technologies democratize many opportunities for participation, especially in entrepreneurial finance through initial coin offerings, crowdfunding, and online IPO auctions (Ritter (2013)). We understand very little about the social efficiency of information aggregation and financing given the large number and decentralized nature of the agents.

To better understand these issues, we introduce AoN design in a standard framework of information cascade à la Bikhchandani, Hirshleifer, and Welch (1992), allowing both endogenous pricing (as in Welch (1992)) and AoN threshold-setting. An proposer a project sequentially approaches N agents who choose to support or reject the project. Each supporter pays a pre-determined contribution, and then gets a payoff normalized to one if the project is good. All agents are risk-neutral and have a common prior on the project's quality. They each receives a private, informative signal, and observes the decisions of preceding agents, before deciding whether to make a contribution. Deviating from the literature, we introduce a decision threshold potentially determined by the proposer—supporters only pay the cost and receive the project payoff if the number of supporters reaches the AoN threshold.

AoN thresholds lead to uni-directional cascades in which agents never rationally ignore positive private signals to reject the project, but may rationally ignore negative private signals to support the project. Consequently, information production becomes more efficient, especially with a large crowd of agents, leading to more successes of good projects and weeding out some bad projects. When the decision threshold and issuance price are endogenously chosen by the proposer of the project, there is less underpricing relative to the standard setting of sequential sales with information cascades (Welch (1992)), which improves financing efficiency. In particular, when the number of agents grows large, equilibrium project implementation and information aggregation approach the first best, in stark contrast to earlier findings that blockages form and information aggregation is imperfect in an environment with information cascades (Banerjee (1992); Lee (1993); Bikhchandani, Hirshleifer, and Welch (1998); Ali and Kartik (2012)).

With any given AoN threshold, we show that before reaching the threshold, the aggregation of private information only stops upon an UP cascade, in which the public Bayesian posterior belief is so positive that agents always support the project regardless of their private signals. The intuition derives from the fact that the threshold partially internalizes the externalities of each individual's action and thus mitigates the herding concern. An agent making decision cares about subsequent agents because their actions determine whether and in what scenarios the threshold can be reached, hence she accounts for the fact that her action reveals information about the project's quality beyond what is contained in her being pivotal for project implementation.

Interestingly, such forward-looking considerations lead to asymmetric outcomes: agents with positive private signals always find it optimal to support because by doing so they essentially delegate their decisions to a subsequent "gate-keeping" agent whose supporting decision brings the support to the threshold. All supporting predecessors benefit from the delegation because the "gate-keeping" agent observes previous actions and is more informed by the time she makes the decision. Meanwhile, agents with negative private signals are reluctant to support even before the threshold is reached, because in equilibrium their actions may mislead subsequent agents and causes either a too-early UP cascade or reaching the AoN threshold without enough number of positive signals, both implying a negative payoff for her. Therefore, DOWN cascades (where agents ignore positive private signals to reject) do not occur because they are all interrupted by agents with positive signals until the threshold is about to be reached. Agents including and following the "gate-keeping" one know that the project would be implemented for sure when they support, and the situation returns to the classical information-cascade setting.

To analyze the optimal choice of AON threshold and its impact on pricing and information production, we next endogenize the proposer's choices of the AoN threshold and price (contribution per agent) to maximize the proceeds (or the amount of support in non-financial scenarios). A higher AoN threshold excludes more DOWN cascades but is also less likely to be reached. We show that the proposer's concern about potential DOWN cascades dominates and wants to completely excludes DOWN cascades, in the same spirit as Welch (1992). The caveat is that the proposer utilizes both price and threshold to achieve this. Consequently, with endogenous issuance pricing, there is no DOWN cascade which stops private information aggregation, and good projects are more likely to be financed.

The exclusion of DOWN cascades has two important implications. First, it allows projects of high quality but with costly production costs to be financed. In the standard information cascade setting, Welch (1992) shows that the proposer endogenously charges a low price to induce an UP cascade from the very beginning and prevent potential DOWN cascades. The feasible price range is limited because the price must be lower than the posterior of the first agent with a positive signal to prevent an immediate DOWN cascade. This limited price range prevents financing high-quality projects with high production costs. Our model demonstrates that an AoN threshold provides the proposer an additional tool to avoid DOWN cascades. On the one hand, a higher price increases the profit the proposer collects from each supporting agent. On the other hand, a higher price sets a higher bar for implementation and delays UP cascades. With the optimal threshold, the project is only implemented when the agents' posterior beliefs improve sufficiently to justify the price.

Therefore the proposer can charge a sufficiently high price to cover the project cost without worrying about DOWN cascades. Uni-directional cascades thus enlarge the feasible pricing range for fundraising. When N is large, financing is efficient in the sense that good projects are always financed while bad projects are never financed.

Second, the exclusion of DOWN cascades also affects information aggregation. In standard models of financial markets with information cascades, the severe underpricing excludes information aggregation by triggering UP cascade from the very beginning and this result is independent of the size of investor crowd. With AoN, information aggregation takes place before an UP cascade. Uni-directional cascades thus partially restore information aggregation. Since the delay of UP cascade is less costly given a large agent base, the proposer facing a large number of potential agents can charge a higher price for issuance to delay potential UP cascades and in so doing producing more information. Full information aggregation is achieved when the crowd base N becomes very large, despite the presence of information cascades.

Furthermore, by aggregating information before investment is sunk, financial activities such as crowdfunding adds an option value to experimentation, which can facilitate entrepreneurial entry and innovation (Manso (2016)). Chemla and Tinn (2016) and Xu (2017) find empirical evidence that entrepreneurs utilize the information produced through crowdfunding for further decision-making. Our model provides a theoretical framework to rationalize and interpret their empirical findings.

Finally, we demonstrate that our key insights apply even when agents have the option to postpone their decisions, and are thus less subject to the usual critiques on information-cascade models. We also analyze the limiting behavior of all equilibrium outcomes to show how our findings are robust to equilibrium selection, and how the AoN design induces in sequential interactions properties of simultaneous-move games, a phenomenon novel to models of information cascades.

Our theory is mainly motivated by and applies to entrepreneurial finance, particularly, crowdfunding. Since its inception in the arts and creativity-based industries (e.g., recorded music, film, video games), crowdfunding has quickly become a mainstream source of capital for early startups, and typically features AoN threshold.[1] Decentralized crowd often chance

[1]In the span of a few years, its total annual volume has reached 34.4 billion USD globally at the dawn of 2017. It has surpassed the market size for angel funds in 2015, and the World Bank Report estimates that

upon a project for example through social media, but lack the expertise to fully evaluate a startup's prospect or a product's quality (due diligence is too costly when their investment is limited), leading to high uncertainty and collective-action problems (Ritter (2013)). Yet the observation of funding targets and supports up-to-date allow them to learn and act in a Bayesian manner (Agrawal, Catalini, and Goldfarb (2011); Zhang and Liu (2012); Burtch, Ghose, and Wattal (2013)). Another example is venture financing: In an angel or A-round of financing, entrepreneurs seek financing from multiple agents. Investors approached later often learn which others indicate support for the project, and many condition their contributions on the fundraising reaching the threshold the entrepreneurs specify or the minimum to implement the project.[2] Yet one more oft-discussed example involves initial public offerings (IPOs): late investors learn from observing the behavior of early investors, and IPOs with high institutional demand in the first days of book-building also see high levels of bids from retail investors in later days (e.g., Welch (1992) and Amihud, Hauser, and Kirsh (2003)). The issuer faces an unknown demand for its stock and aggregates information from sequential agents about the demand curve (e.g., Ritter and Welch (2002)), therefore the issuer may choose to withdraw the offering if the market reaction is lukewarm.[3] Our findings highlight the importance of designing AoN thresholds and utilizing new technologies

global investment through crowdfunding will reach $93 billion in 2025 ($http://www.infodev.org/infodev-files/wb_crowdfundingreport-v12.pdf$) The US deregulation also passed the law to allow non-accredited agents to join equity-based crowdfunding, further fueling the development. Specifically, on April 5, 2012, President Obama signed into law the Jumpstart Our Business Startups (JOBS) Act. Adding to then extant donation and reward based crowdfunding platforms, the JOBS Act Title III legalized crowdfunding for equity by relaxing various requirements concerning the sale of securities to non-accredited investors in May 2016 (Title II already permits accredited investors). What is more, with the rise of initial coin offerings, alternative corporate crowdfunding emerges, with over two billion dollars raised in the US in the first half of 2017. The Crowdfund Act also indicates that AoN feature will likely be mandated, because intermediaries need to ensure that all offering proceeds are only provided to the issuer when the aggregate capital raised from all agents is equal to or greater than a threshold offering amount, and allow all agents to cancel their commitments to invest, as the Commission shall, by rule, determine appropriate (Sec. 4A.a.7). See http: // beta.congress .gov / bill / 112th- congress / senate- bill / 2190 / text.

[2]For example, the blockchain startup String Labs approached multiple agents such as IDG capital and Zhenfund sequentially, many of whom decided to invest after observing Amino Capital's investment decision, and conditioned the pledge on the founders' "successfully fundraising" in the round (meeting the AoN threshold). Syndicates involving both incumbent agents from earlier rounds and new agents are also common.

[3]With limited distribution channels by investment banks, it takes the underwriter times to approach interested agents, who are typically institutions that do not communicate amongst one another. Strong initial sales encourage subsequent support while slow initial sales discourage subsequent investing. During an IPO, the issuer may decide to not continue with its proposed offering of securities if he observes a poor agent interest. IPO is therefore also characterized by sequential arrival and AoN. In both Internet-based crowdfunding and IPO, there is no market for agents to trade, and prices are fixed by entrepreneurs or the underwriter.

to reach out to a broader base of potential supporters, not only for the entrepreneur but also for welfare considerations.

Literature — Our paper foremost contributes to the large literature on information cascades, social learning, and rational herding (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1998; Chamley, 2004). Our model largely builds on Bikhchandani, Hirshleifer, and Welch (1992) which discuss informational cascade as a general phenomenon. Welch (1992) relates information cascade to IPO underpricing. Studies such as Anderson and Holt (1997), Çelen and Kariv (2004), and Hung and Plott (2001) provide experimental evidence for information cascades. We add to the literature by incorporating and endogenizing AoN threshold as a common form of payoff interdependence, and show that it mitigates inefficiencies typically associated with information cascades. Moreover, we also contribute by characterizing multiple equilibrium outcomes for information cascades with strategic complementarity.

Information aggregation has been a central question in information economics (e.g., Wilson (1977) and Pesendorfer and Swinkels (1997), and Kremer (2002)). In the context of sequential voting, Ali and Kartik (2012) explore the optimality of collective choice problems; Dekel and Piccione (2000) extend Feddersen and Pesendorfer (1997) and demonstrate the equilvalence for simultaneous and sequential elections. But in our setting, sequential action matters because it reveals more information than is contained in the event that the voter is pivotal. The payoff structure in our setting is closer to the classical models of information cascades in that it not only depends on whether the project is implemented, but also depends on whether the agent supports the project. Therefore, our model describes a separate set of phenomena from voting, and the sincere behavior in Ali and Kartik (2012) (also Ali and Kartik (2006)) only applies after AoN is reached. Moreover, most information cascade models predict imperfect information aggregation even in large markets (e.g., Ali and Kartik (2012)) because the action set typically does not map to each agents posterior belief one-to-one (Lee (1993)). We show that even absent such requirements, endogenous AoN thresholds can still achieve full aggregation—a novel result in models allowing information cascades.

The paper also adds to an emerging literature on AoN design in the context of crowdfunding. Strausz (2017) and Ellman and Hurkens (2015) find that AoN is crucial for mitigating moral hazard and price discrimination. Chemla and Tinn (2016) demonstrate that the AoN design Pareto-dominates the alternative "keep-it-all" (KiA) mechanism. Chang (2016) also shows that under common-value assumptions AoN generates more profit by making the expected payments positively correlated with values. Instead of introducing moral hazard or financial constraint, or derives optimal designs in static settings, we focus on pricing and information production, especially under endogenous AoN arrangements and with dynamic learning.[4] Regarding information efficiency, Brown and Davies (2017) shows that an exogenous AoN threshold can adversely affect the financing efficiency in a static setting. Also closely related is Hakenes and Schlegel (2014) which argues that endogenous loan rates and AoN thresholds encourage information acquisition by individual households in lending-based crowdfunding. We consider both exogenous and endogenous AoN thresholds in a dynamic environment and demonstrate gains in informational efficiency as well as in financing efficiency.

The rest of the paper is organized as follows: Section 2 sets up the model and derives agents' belief dynamics; Section 3 characterizes the equilibrium, starting with exogenous AoN threshold and issuance price to highlight the main mechanism of uni-directional cascades, before endogenizing them; Section 4 discusses model implications and demonstrates how AoN better utilizes the wisdom of the crowd to improve financing and information-production efficiencies; Section 5 discuss option to wait and limiting behaviors of all equilibria; Section 6 concludes. All proofs are in the appendix.

2 A Model of Directional Cascades

2.1 Setup

Consider a project (or proposal or type of behavior) presented to a sequence of agents $i = 1, 2, \ldots, N$ who can support (adopt) or reject. The action of agent i is $A_i \in \{S, R\}$, where S denotes supporting and R, rejecting.[5] If the proposal is implemented eventually, then every supporting agent incurs a supporting contribution c (which is a cost to the agent), and

[4] Regarding security design in crowdfunding, Li (2017) argues how profit-sharing contracts could be optimal and Hildebrand, Puri, and Rocholl (2016) provide evidence of perverse incentives in debt crowdfunding using data from Prosper.

[5] While in crowdfunding they may choose the quantity of investment, under risk-neutrality it suffices to consider the case where each supporter can only make a unit contribution: If an agent finds the project to be positive NPV, then she scales to full investment capacity. What matters is the information conveyed by her action. In practice, crowdfunders often observe both the total capital raised and the number of supporters thus far (Vismara (2016)).

receives the benefit V, which is either 0 or 1. In scenarios such as petition-signing or fashion adoption, c is the signing or adoption cost. In fund-raising activities such as crowdfunding or venture financing rounds, c is the amount of money that each supporting agent pays and is often pre-determined by the proposer.[6]

All agents including the proposer are rational, risk-neutral, and share the same prior that the project type can be either $V = 0$ and $V = 1$ with equal probability.[7] Each agent i observes one conditionally independent private signal $X_i \in \{H, L\}$. Signals are informative in the following sense:

$$Pr(X_i = H|V = 1) = Pr(X_i = L|V = 0) = p \in (\frac{1}{2}, 1); \tag{1}$$

$$Pr(X_i = L|V = 1) = Pr(X_i = H|V = 0) = q \equiv (1 - p) \in (0, \frac{1}{2}). \tag{2}$$

We depart from the literature by incorporating the observed "all-or-nothing" (AoN) scheme into this setup: the proposer receives "all" if the campaign succeeds in reaching a pre-specified threshold number of supporters, and "nothing" if it fails to do so. In other words, the project is implemented if and only if more than T_N agents support, where T_N could be exogenous in the case of legal legacy, or endogenous in the case of IPO issuance or crowdfunding. We treat c and AoN threshold as exogenous in this section to first understand the universal impact of the AoN scheme, before endogenizing them in Section 3.2.

[6]There is a separate literature studying herding and financial markets that allows asset price to dynamically change and focuses on asset pricing implications (e.g.,Avery and Zemsky (1998), Brunnermeier (2001), Vives (2010), and more recently Park and Sabourian (2011). We follow the standard cascade models to fix the price for taking an action ex ante, which more closely matches applications such as those in entrepreneurial finance.

[7]Idiosyncratic preferences or private valuations would not change the intuition of the economic mechanism. Our specification is fitting for equity-based crowdfunding. Even in reward-based crowdfunding whereby agents have private valuations and preferences, there is a common value corresponding to the basic quality of the product. Our assumption of common value also allows us to make unambiguous welfare comparisons concerning the financing and information aggregation efficiency (e.g., Fey (1996) and Wit (1997)).

Agents' Information and Decision

The order of agents' decision-making is exogenous and known to all.[8] This is equivalent to observing both supporting and rejecting actions of previous agents, a standard assumption in the literature on information cascades.[9] In other words, when agent i makes her decision, she observes her own private signal X_i and decisions made by all those ahead of her, that is, $\{A_1, A_2, \ldots, A_{i-1}\}$. Agents Bayesian update their beliefs using private signals and inferences from the observed actions of their predecessors in the sequence. Let $\mathcal{H}_i \equiv \{A_1, A_2 \ldots, A_i\}$ be the action history till agent i's turn, and N_S be the ultimate total number of supporting agents. Agent i's problem is:

$$\max_{A_i} \ \mathbb{E}\left(V - c | X_i, \mathcal{H}_{i-1}, N_S \geq T_N\right) \mathbb{1}_{\{A_i = S\}}, \tag{3}$$

where $\mathbb{1}_{\{A_i=S\}}$ is the indicator functions for supporting decision. If $\mathbb{E}\left(V | X_i, \mathcal{H}_{i-1}, N_S \geq T_N\right) > c$, supporting is agent i's optimal choice. When $\mathbb{E}\left(V | X_i, \mathcal{H}_{i-1}, N_S \geq T_N\right) = c$, we assume that:

Assumption 1 (Tie-breaking)**.** *When indifferent between supporting and rejecting, an agent supports if the AoN threshold can be reached with all remaining agents supporting.*

In other words, whenever indifferent in terms of payoff consideration, an agent supports the project if conditional on all subsequent agents' supporting, threshold T_N can be reached. As we discuss later in Section 5, the strategic complementarity of agents' actions becomes important under AoN threshold. What this assumption rules out are the trivial equilibria where everyone believes that there would not be enough supporting agents and therefore coordinates to not supporting. The assumption is also non-restrictive in the sense that when

[8]While real world examples such as crowdfunding may involve endogenous orders of agents, our setup allows us to relate and compare to the large literature on information cascades which typically has exogenous orders of agents. We show in Section 5.1 that our fundamental result is robust when agents have options to wait. Related is Liu (2018) that studies how AoN affects the timing of investor moves.

[9]In the application in crowdfunding, this is equivalent to observing fund raised to-date (and time) and knowing the starting time of fundraising and the agent arrival rate.This is a common assumption in crowdfunding studies (e.g., Vismara (2016)). For example, one can check the website for visitor/viewer flow, and infer how many have supported or rejected per unit time. Guarino, Harmgart, and Huck (2011) and Herrera and Hörner (2013) consider information cascades when only one of the binary actions is observable, and find that welfare could improve and cascades can be uni-directional. Unlike these studies, our results do not rely on the assumption that agents' position is unknown, and we study the consequence of both exogenous and endogenous AoN thresholds under the standard cascade setting.

the proposer endogenously sets c, he can always lower c by an arbitrarily small amount, possibly through a subsidy, to break the tie and induce the support.

We note that the binary information and action structure here is the canonical focus in both the information cascade literature (Bikhchandani, Hirshleifer, and Welch (1992)) and the voting literature (Feddersen and Pesendorfer (1996) and McLennan (1998)). Moreover, the nature of the signal is not crucial to the results and intuition because binary action implies that others can observe at most a binary partition of the agents signal space (Gale (1996)). The discreteness of the action set, especially that the agents have the option to reject, is important for producing information cascades (Gul (1993)). When the condition fails, there would not be cascades. But information aggregation nevertheless slows down (Vives (1993)), and AoN can potentially help increasing financing and information aggregation efficiency.

proposer's Optimization

Let $0 \leq \nu < 1$ be the per issuance cost for the proposer. In the context of reward-based crowdfunding, ν could be the production cost of each product. In the equity-based crowdfunding or IPO process, ν can be interpreted as the issuer's share reservation value. In essence, varying ν is equivalent to varying the prior on the project NPV to a social planner. The proposer chooses price c and AoN threshold T_N to solve the following problem:

$$\max_{c,T_N} \pi(c, T_N, N) = E[(c-\nu)N_S \mathbb{1}_{\{N_S \geq T_N\}}], \tag{4}$$

where $\mathbb{1}_{\{N_S \geq T_N\}}$ is the indicator function for project implementation. In fund-raising scenarios, the proposer tries to maximize his expected profit. In non-financial scenarios, c can be interpreted as the amount of support from each agent and the proposer solicits the maximum amount of support.

2.2 Belief Dynamics and Information Cascades

We first analyze the dynamics of the common posterior belief after observing the action history. Note that agents only observe the history of actions, so the "signals" below are really inferred signals.

Lemma 1. *Given a series of signals $X \equiv \{X_1, X_2, \ldots, X_n\}$, the ratio of the posterior probability of $V = 1$ to that of $V = 0$ is*

$$\frac{Pr(V=1|X)}{Pr(V=0|X)} = \frac{p^K}{q^K},$$

where $K \equiv \#of\ H\ signals - \#of\ L\ signals$.

Lemma 1 states that the posterior belief of project type only depends on the difference between numbers of H and L signals so far, a convenient property also in Bikhchandani, Hirshleifer, and Welch (1992). Given Lemma 1, an agent's expected project value conditional on there bing $K = k$ more H signals (inferred from her own signal and previous agents' actions) is then,

$$V_k \equiv \mathbb{E}(V|K=k) = \frac{p^k}{p^k + q^k}. \tag{5}$$

It is apparent that the expected project payoff is strictly monotonically increasing in k.

When agents act regardless of their private signals, the market fails to aggregate dispersed information. Our notion of informational cascade follows the literature standard (e.g. Bikhchandani, Hirshleifer, and Welch (1992)).

Definition 1 (Information Cascade)**.** *An information cascade occurs if a subsequent agent's action does not depend on her private information signal. An* UP *cascade occurs if each subsequent agent supports the project regardless of her private signal. A* DOWN *cascade occurs if she rejects the project regardless of her private signal.*

Notice that we have taken the convention of calling it a cascade as long as the NEXT agent and the ones after ignore their private information, even though the current agent may still use private signal. This simplifies exposition in the proof. In classic cascades models, both UP and DOWN cascades are possible. If a few early agents observe H signals, their contributions may push the posterior so high that the project remains attractive even with a private L signal. Similarly, a series of L signals may doom the offering. An early preponderance towards supporting or rejecting causes all subsequent individuals to ignore their private signals, which are then never reflected in the public pool of knowledge.

It remains to specify how agents infer K from the history of actions $\mathcal{H}$ up till their time of decision-making. An action is either informative or non-informative in equilibrium. If it is informative, agents use Bayes rule to infer the signal of a previous agent from her action; if an action is non-informative, remaining agents' beliefs stay the same. It is then clear that the mapping from $\mathcal{H}$ to K depends on the equilibrium definition, which we specify next.

2.3 Equilibrium Definition

We use the concept of perfect Bayesian Nash equilibrium (PBNE). Before an information cascade starts, we call an agent a "free-rider" if everyone in equilibrium believes her action is uninformative and ignores it; otherwise, we can the agent an "informer". In our baseline model we focus on "informer equilibria" without "free-riders", defined as:

Definition 2 (Informer Equilibrium)**.** *An informer equilibrium consists of proposer's proposal design $\{c^*, T_N^*\}$, agents' action strategies $\{A_i^*(X_i, \mathcal{H}_{i-1}, c^*, T_N^*)\}_{i=1,2\ldots,N}$, and their belief such that:*

1. *For each agent i, given the required contribution c^* and T_N^*, associated T_N^* and other agents' investment strategies $\{A_j^*(X_j, H_{j-1}, c^*, T_N^*)\}_{j=1,2,\ldots,i-1,i+1,\ldots,N}$, investment strategy $A_i^*(X_i, \mathcal{H}_{i-1}, c^*, T_N^*)$ solves her optimal problem:*

$$A_i^* \in \text{argmax } [\mathbb{E}\left(V - c|X_i, \mathcal{H}_{i-1}, N_S \geq T_N\right)]\mathbb{1}_{A_i=S}; \tag{6}$$

2. *Given investment strategies $\{A_i^*(X_i, \mathcal{H}_{i-1}, c^*, T_N^*)\}_{i=1,2\ldots,N}$, c^* and T_N^* solve proposer's problem:*

$$\{c^*, T_N^*\} \in \text{argmax} \quad \pi(c, T_N, N). \tag{7}$$

3. *Agents Bayesian-update their beliefs, knowing that everyone is an informer before an information cascade.*

Point 3 is what makes the equilibrium an "informer" equilibrium because before an information cascade, agents' updates their beliefs from observing preceding agents' actions.

We focus on the concept of informer equilibrium because it is natural and clearly illustrates our key mechanisms and economic intuition. Moreover, we discusses all other PBNE Section 5.2, and show that they are, if exist, simply variants of "informer equilibrium" that behave exactly the same in the limit of large agent base.

2.4 Benchmark of Standard Cascade without AoN Threshold

If there is no AoN (or equivalently, $T_N = 1$), then for each agent, her payoff does not depend on what later agents do. Thus, the equilibrium is essentially the same as the one characterized in Bikhchandani, Hirshleifer, and Welch (1992) and Welch (1992). That is, each agent i chooses to support if and only if

$$\mathbb{E}(V|X_i, \mathcal{H}_{i-1}) \geq c. \tag{8}$$

In this equilibrium, both UP and Down cascades can occur. The aggregation of public information stops once a cascade arrives. Information cascade also affect pricing policy. The following proposition shows that without AoN threshold, the contribution is under-priced when the precision of private signals is not high enough.

Lemma 2. *The proposer always charges $c \leq p$. In particular, when $\nu = 0$ and $p \leq \frac{3}{4} + \frac{1}{4}(3^{\frac{1}{3}} - 3^{\frac{2}{3}})$, the optimal contribution is $c^* = 1 - p < \frac{1}{2} = \mathbb{E}(V)$.*

The lemma is basically a restatement of the underpricing result in Welch (1992), especially Theorem 5. The first general pricing upper bound comes from the concern for potential DOWN cascades. If proposer charges $c > p$, then even with a H signal, the first agent chooses rejection and so does every subsequent agent, leading to a DOWN cascade starts at the very beginning, which yields 0 benefit for sure.

The second result concerns optimal pricing when the individual signal is not very precise and cascades are a relevant issue. Note that $\nu = 0$ is the case in Welch (1992). UP and DOWN cascades affect the proposer's payoff asymmetrically even though they both reduce the information aggregation among agents. While the proposer benefits from UP cascades by attracting contributions from late agents with L signals, he is concerned with DOWN cascades since a few early rejections may doom the offering. As discussed in Bikhchandani, Hirshleifer, and Welch (1992), the impact of cascades largely depends on the precision of the private information. If the information is precise, then cascades is not a big concern since a cascade only occurs when the aggregated public information is sufficiently informative to dominate one's private signal, suggesting a high probability of correct cascades. When the private information precision is low, the concern of DOWN cascades pushes down the price to a level such that an UP cascade starts at the very beginning with probability 1. Because

$c^* < \mathbb{E}[V]$, the optimal pricing entails underpricing ex ante so that the first agent finds it attractive even with a L signal. To be clear, depending on the true project quality, we may still have ex-post overpricing when $V = 0$.

3 Equilibrium Characterization

We derive the equilibrium in several steps. First, we take the price c and AoN threshold T_N as exogenous, and examine the sub-game equilibrium of agents' decisions on whether to support or reject the project. We then solve for the proposer's endogenous pricing and AoN-threshold setting, and compare the equilibrium outcomes to those in the benchmark case without AoN thresholds.

3.1 Exogenous Price and AoN Threshold

The first main result in our paper is to show that with the AoN feature, there exists an equilibrium such that before the AoN threshold is about to be reached, only UP cascades may exist.

Proposition 1. *For any given pair $\{c, T_N\}$, there exists an unique informer equilibrium such that:*

1. *When there are at least $T_N - 1$ supporting predecessors:*
   - *The current agent i chooses to support if and only if*

$$\mathbb{E}[V|X_i, \mathcal{H}_{i-1}] \geq c. \tag{9}$$

2. *When there are strictly less than $T_N - 1$ supporting predecessors:*
   - *Agent i with signal H always supports the project;*
   - *Agent i with signal L supports if and only if:*

$$\mathbb{E}_i[V|K = k - 1] \geq c, \tag{10}$$

where the subscript i indicates the ith agent's expectation at the time of decision making.

For notational simplicity, we drop the subscript i in the remainder of the paper unless otherwise specified. Proposition 1 describes adoption strategies for agents.

The proposition implies that there is no DOWN cascade before approaching the AoN threshold. In Part 1 of the proposition, when the AoN threshold would be reached with one more supporting agent (there are at least $T_N - 1$ supporting predecessors), the current agent knows that the project would be implemented if she supports, and she faces exactly the same optimization problem as in standard cascade model. However, in Part 2 of the proposition before the AoN threshold is approached (there are strictly less than $T_N - 1$ supporting predecessors), in the equilibrium agents with H signals always support regardless of the history they observe while agents with bad signals support only when there is an UP-cascade. To see this, we note that an agent with a H signal and having observed strictly less than $T_N - 2$ supporting predecessors has protection on her investment because she does not need to pay if the project turns out to be bad. A subsequent agent observing $T_N - 1$ supporting predecessors would be the "gate-keeper" for her because they share the same interests but the subsequent agent observes a longer history.

To complete the reasoning, we need to show that, when there is no UP cascade yet, agents with bad signals have no incentive to deviate to support so observing a longer history is helpful. Agents with bad signals does not deviate because all subsequent agents would misinterpret her action and form wrong posterior beliefs, and the over-optimistic belief implies that they either start an UP cascade too early or reach the AoN threshold when the true posterior is not high enough. Taking that into account, agents with bad signals find deviation unattractive. Again, when there are at least $T_N - 1$ preceding supporting agents, follow agents know that the project would be implemented for sure when they invest, and their optimal adoption decision problem is exactly the same as in standard information cascade models, and both UP and DOWN cascades are possible.

The proof for Proposition 1 suggests both the possibility and arrival time of cascades, as summarized in the following corollary.

Corollary 1. *When $c \in (V_{k-1}, V_k]$, an UP cascade starts whenever there are $k + 1$ more agents supporting rather than rejecting, a DOWN cascade starts whenever there are $k-2$ more* *agents supporting rather than rejecting and there are at least $T_N - 1$ supporting predecessors.*

One can interpret UP cascades as the source of type I error in information aggregation since it may falsely accept the project when it is bad. On the other hand, DOWN cascades introduce type II error, rejecting the proposal when it is actually good. Intuitively, if the agent base and corresponding AoN threshold is large, DOWN cascades do not occur and the type II error completely disappears, that is to say, all good project are implemented, as summarized in the next proposition.

Proposition 2. *If $\lim_{N\to\infty} \frac{T_N}{N} \in (0,1)$, then as $N \to \infty$, a good project with $V = 1$ is implemented almost surely with an UP cascade.*

3.2 Endogenous Price and AoN Threshold

In real life, especially in fundraising activities such as crowdfunding and venture financing, the proposer endogenously set the price of each contribution and the AoN threshold to maximize his revenue. We now endogenize the proposer's decision by solving the stage where the proposer decides on the price and AoN threshold before other agents make decisions. Our findings are important because the underpricing or overpricing of securities or products may affect the success or failure of the issuance, and thus directly impact the real economy. IPOs with limited distribution channels of investment banks (Welch (1992)) constitute a salient example.

Without loss of generality, we focus on the case of $0 \leq \nu \leq V_N$. If $\nu > V_N$, then the marginal production cost is higher than the highest possible posterior, and the proposer charges $c = \nu$ and gets zero profit. Similar to Proposition 1, with the AoN feature, there exists an informer equilibrium such that only UP cascades may exist.

Proposition 3. *There is an informer equilibrium when the investment contribution (price) $c^* \in (0,1)$ and the AoN threshold $T_N^* \leq N$ are endogenous, such that:*

1. *$c^* \in \{V_k | k = -1, 0, 1, \cdots, N\}$;*

2. *Let $\mathbb{E}(V|x, N)$ be the posterior mean of V given there are x number of H signals out of N observations, then*

$$\mathbb{E}(V|T_N^*, N) \leq c^* < \mathbb{E}(V|T_N^* + 1, N); \tag{11}$$

3. *Agents with signal H always support the project;*

4. *Agent i with signal L contributes if and only if:*

$$\mathbb{E}(V|K = k - 1) \geq c^*. \tag{12}$$

Proposition 3 characterizes agent strategies and the proposer's endogenous proposal design in the equilibrium. As we discuss in Section 5, all other possible equilibria are variants of the equilibrium characterized here, and our main results remain. We therefore focus on the equilibrium described in Proposition 3.

The proof for Proposition 3 again rules out DOWN cascades, and suggests both the possibility and arrival time of UP cascades. We summarize their characterization here:

Corollary 2. *In the equilibrium characterized in Proposition 3, there would be no DOWN cascades. If $c \in (V_{k-1}, V_k]$, an UP cascade starts whenever there are $k + 1$ more agents supporting rather than rejecting.*

In the equilibrium the proposer chooses the optimal level of AoN threshold jointly with price to exclude DOWN cascades. Recall that there is no DOWN cascade before approaching the threshold. A higher AoN threshold reduces the burden of using underpricing to exclude DOWN cascade once the threshold is reached. Yet a higher threshold itself is more difficult to reach. In the equilibrium, the proposer considers the tradeoff and for a given c chooses the lowest level of AoN threshold to completely exclude DOWN cascades.

Next, we examine the informational environment in such an UP cascaded equilibrium, and its pricing implications. Lemma 1 and Equation (5) show that the posterior only depends on K, the difference between numbers of H and L signals. If the price is V_{k-1}, then an UP cascade starts once $K = k$. Since each agent privately observes either H or L and in the equilibrium her decision perfectly reveals her private signal before an UP cascade starts, the arrival of an UP cascade is equivalent to the first passage time of a one-dimension biased random walk. The following lemma is based on Van der Hofstad and Keane (2008), and lays the foundation for our analysis on the distribution of UP cascades' arrival time.

Lemma 3 (Hitting Time Theorem)**.** *For a random walk starting at $k_0 \geq 1$ with i.i.d. steps $\{Y_i\}_{i=1}^{\infty}$ satisfying $Y_i \geq -1$ almost surely, the distribution of the stopping time $\tau_0 = \inf\{n :$*

$S_n = k_0 + \sum_{i=1}^{n} Y_i\}$ *is given by*

$$Pr(\tau_0 = n) = \frac{k_0}{n} Pr(S_n = 0). \tag{13}$$

To characterize the distribution of UP cascades arrival time, let $\varphi_{k,i}$ be the probability that an UP cascade starts at agent i, then

Lemma 4. *If $c \in (V_{k-2}, V_{k-1}]$, then the probability that an UP cascade starts at agent i is*

$$\varphi_{k,i} = \frac{k}{i} \left(\begin{array}{c} i \\ \frac{i+k}{2} \end{array} \right) (pq)^{\frac{i-k}{2}} \frac{p^k + q^k}{2}, \tag{14}$$

where

$$\left(\begin{array}{c} i \\ \frac{i+k}{2} \end{array} \right) = \begin{cases} \frac{i!}{\frac{i+k}{2}!\frac{i-k}{2}!} & \text{if } i \geq k \text{ and } k+i \text{ even;} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Since for any $c \in (V_{k-1}, V_k]$, all agents make the same investment decisions, the proposer can always charge $c = V_k$ and receives a higher profit. Without loss of generality, we focus our pricing analysis on $c \in \{V_{-1}, V_0, \ldots, V_N\}$. We exclude cases for $k < -1$ because $V_{-1} = 1 - p$ is low enough to induce an UP cascade from the very beginning for sure.

Now we consider the optimal pricing. An UP cascade only occurs when the posterior given another L signal is higher than c, and all subsequent agents support the project. The project is eventually implemented once an UP cascade starts. Meanwhile, for any agent $i \leq N - 2$, if the UP cascade has not started yet, then there is a strictly positive possibility that the project will not be implemented. So a project is eventually funded if and only if the following condition holds: either (1) there is an UP cascade, or (2) agent N supports the project and the total number of supporting agents is exactly T_N. In either cases, we can compute the profit associated with c, as formalized in Proposition 4. Before going there, we illustrate the two scenarios in Figure 1, which plots the difference between supporting agents and rejecting agents when n agents have arrived. The figure also includes a sample path that leads to funding failure because AoN threshold is not reached.

Proposition 4. *When the price is $c = V_{-1} = 1 - p$, the proposer's expected profit is* $(1 -$

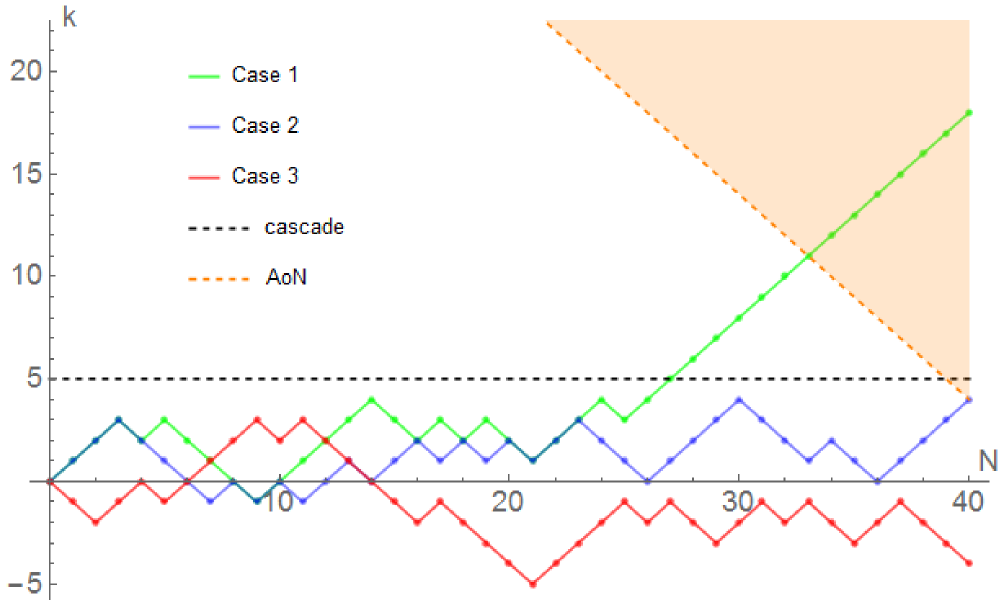

Figure 1: Evolution of Support-Reject Differential

Simulated paths for $N = 40$, $p = 0.7$, $c^* = V_5 = 0.9673$, and AoN threshold $T^*(N) = 22$. Case 1 indicates a path that crosses the cascade trigger $k = 5$ at the 26th agent and all subsequent agents support regardless of their private signal; case 2 indicates a path with no cascade, but the project is still funded by the end of the fundraising; case 3 indicates a path where AoN threshold is not reached and the project is not funded. The orange shaded region above the AoN line indicates that the project is funded.

$p-\nu)N$. Given a price $c = V_{k-1}$, $k \in \{1, 2, \dots, N\}$, the proposer's expected profit is

$$\pi(V_{k-1}, N) = \begin{cases} (V_{k-1}-\nu)\left[\sum_{i}^{N}\varphi_{k,i}(N-\frac{i-k}{2}) + \frac{p^{k-1}q+pq^{k-1}}{p^k+q^k}\varphi_{k,N}\frac{N+k-2}{2}\right] & \text{if } k+N \text{ even;} \\ (V_{k-1}-\nu)\left[\sum_{i}^{N-1}\varphi_{k,i}(N-\frac{i-k}{2}) + \frac{p^k+q^k}{p^{k+1}+q^{k+1}}\varphi_{k,N+1}\frac{N+k-1}{2}\right] & \text{if } k+N \text{ odd.} \end{cases} \tag{16}$$

Let $k_\nu \in \{0, 1, 2, \dots\}$ be the smallest integer satisfying $V_{k_\nu} \geq \nu$. For each $k \in \{k_\nu, k_\nu+1, k_\nu+2, \dots\}$, there exists a finite positive integer $\underline{N}(k)$ such that for $\forall\ N \geq \underline{N}(k)$, $\pi(V_k, N) > \pi(V_{k-1}, N)$.

Proposition 4 gives an explicit characterization of proposer's expected profit as a function of price V_k and number of potential agents N. Figure 2 provides an illustration on how the profit depends on c.

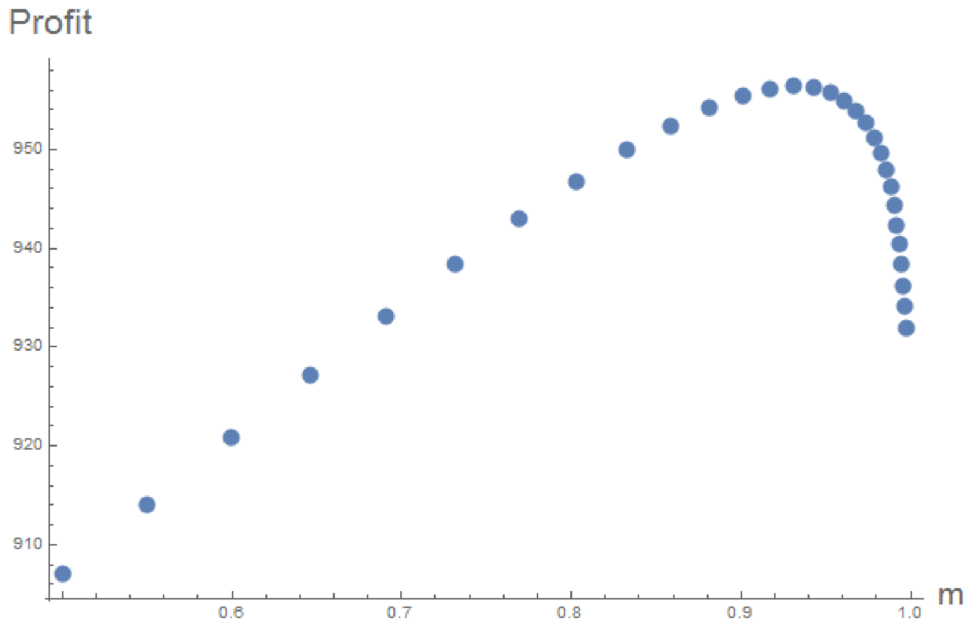

Figure 2: Optimal Pricing: An Illustration with $N = 2000$, $\nu = 0$ and $p = 0.55$.

3.3 Pricing and Large Agent Base

Relative to the benchmark case without AoN threshold, we show that the AoN threshold changes both the pricing upper bound and the underpricing results. More importantly, the result on $\underline{N}(k)$ suggests that, different from Lemma 2, the optimal price depends on the number of potential agents N. A financial technology (Internet-based platforms) that can allow us to reach a greater N thus has a fundamental impact. In the standard cascades models, a DOWN cascade hurts the proposer significantly because subsequent agents all reject. The concern for DOWN cascades pushes down the optimal price, and can cause immediate start of an UP cascade, *independent of the number of agents* because the decisions of later agents have no impact on the first agent's payoffs (Welch (1992)). With the AoN threshold, in the equilibrium there would be no DOWN cascades and one early rejection is not a big concern since agents with H signals would still support the project. Those supporting agents may trigger an UP cascade later, especially when there are many potential agents in the market. The following corollary shows the increasing trend of optimal price c^* as the number of potential agents N grows.

Corollary 3. *For $\forall\ V_k$, there exists a a finite positive integer $N_\pi(V_k)$ such that for $\forall\ N \geq N_\pi(V_k)$, $c^* > V_k$.*

This corollary has a novel implication: as we reach out to more and more agents through technological innovations such as the Internet, the proposer can charge a higher price, and even "overprice" as N becomes big. The left panel in Figure 3 shows the optimal starting point of UP cascades (kth agent) for different values of N, and right panel plots the optimal pricing as a function of N. We note that $c > \mathbb{E}[V]$ in these cases.

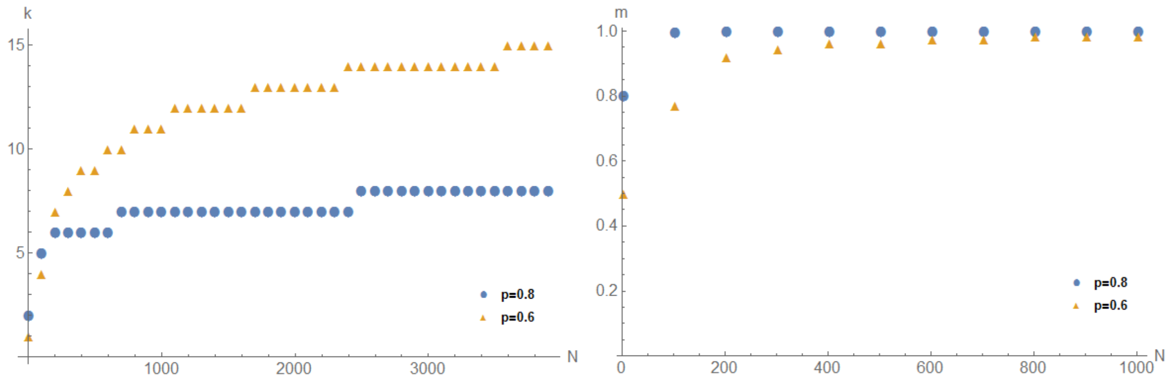

Figure 3: Cascades and optimal prices as N increases

Because for any finite integer $N \geq 2$, $c^*(N) \in \{-1, 0, 1, \ldots, N\}$. Corollary 3 implies that c^* shows an increasing trend in N. Since V_k is a monotonic increasing function in k and $\lim\limits_{k\uparrow\infty} V_k = 1$, it is straightforward to see that

Corollary 4. $\lim_{N\to\infty} c^*(N) = 1$

That is to say, when there is a large base of potential agents, the optimal price approaches the highest possible value, leading to unconditional "overpricing" rather instead of the underpricing in Welch (1992). However, since all agents are rational, conditional on the project is implemented, the price must be lower than the posterior. In other words, high price suggests a corresponding high posterior for project implementation. The unconditional "overpricing" is justified by the information aggregation and associated efficient project implementation.

These corollaries enable us to derive one of key results of the paper: as the agent base grows larger, financing efficiency (in terms of project selection and implementation) and information efficiency converge to the first best despite the presence of information cascades.

Proposition 5. *As $N \to \infty$, good projects are implemented almost surely and bad projects are rejected almost surely.*

Achieving full efficiency in models of information cascades is extremely rare, and one critique on the wisdom of the crowd is precisely the herding behavior. Yet with AoN threshold and a large agent base, the validity of wisdom is restored.

4 Financing and Information Aggregation

We next discuss the implications of AoN scheme for financing feasibility and information aggregation, both of which are also key functionalities of financial markets and platforms. We first demonstrate how the AoN scheme fundamentally changes the feasibility of financing. We then show how informational efficiency improves due to the better harnessing of the wisdom of the crowd, and characterize the resulting informational environment. Finally, we discuss the proposer's real option value from information aggregation, allowing the proposer to carry out the project even if the threshold is missed, or to give up the project even if the threshold is met.

4.1 Feasibility of Fundraising

From Lemma 2, we see that in standard cascade setup there is a pricing upper bound in order for the fundraising or offering to be feasible. This bound becomes a serious concern when the cost ν is non-zero. In particular, any project with a high cost requires charging a high price to cover the cost, thus triggering a DOWN cascade and financing failure for sure. In our model the proposer can still charge a high price and is able to implement the project when aggregated information is good. The following proposition is immediate.

Proposition 6. *Without AoN, no project with $\nu > p$ is financed and information aggregation is infeasible; committing to an AoN threshold enables fundraising and information aggregation even when $\nu > p$.*

The exclusion of DOWN cascades has an important impact on the pricing upper bound, and hence the availability of finance. With AoN threshold, any price $c < 1$ is possible and there would be a strictly positive possibility that the project would be financed given there is a large enough potential agent base. Moreover, from Proposition 2 we know that the good type of project ($V = 1$) will be financed almost surely as the number of agents goes to infinity. In this sense, AoN threshold allows dynamic learning that reduces underpricing, and drives the discrete jump in the feasibility of financing good projects with high production costs. As a result, crowdfunding and the like can enable financing of projects of higher production costs that the proposer cannot avoid DOWN cascades when facing a smaller group of experts, consistent with empirical findings in Mollick and Nanda (2015).[10]

4.2 Wisdom of the Crowd and Welfare

Even when the fundraising is feasible, the process produces little information in most extant models of information cascade. For example, in Welch (1992), cascade always starts from the very beginning, and no private signals are aggregated because once a cascade starts, public information stops accumulating. Nor does the public pool of knowledge have to be very informative to cause individuals to disregard their private signals. As soon as the public pool becomes slightly more informative than the signal of a single individual, individuals defer to the actions of predecessors and a cascade begins.

With AoN threshold, however, the downside risk is removed, and optimal pricing does not necessarily result in information cascades from the very beginning (Lemma 4). Therefore, as long as $c^* > 1-p$, the fundraising also aggregates some private information from the agents, allowing us to harness the wisdom of the crowd to some extent.

What is more, from Lemma 4, the probability that a cascade is correct (UP cascade when $V = 1$) is given by

$$Pr(V = 1|C_i) = \frac{p^k}{p^k + q^k}\mathbb{I}_{\{i \geq k \& k+i \text{ is even}\}}$$

where C_i indicates cascade at i^{th} agent, and k satisfies $V_{k-2} < c \leq V_{k-1}$. Because k is weakly increasing in the pricing c and the optimal pricing is weakly increasing in N (Proposition 4), the following proposition ensues.

Proposition 7. *A cascade starts weakly later with higher price c, and thus with a larger crowd (larger N) when pricing is endogenous. The probability of a cascade being correct is increasing in p, weakly increasing in the pricing c, and weakly increasing in N when pricing is endogenous.*

[10] Mollick and Nanda (2015) find that of the projects that experts and the crowd do not agree on investment decisions, 75% are crowdfunded rather than the other way round.

AoN reduces underpricing, which in turn delays cascade and increases the probability of correct cascades. More importantly, whereas N does not matter in standard cascade models, AoN links the timing and correctness of cascades to the size of the crowd. With a large N as is the case for Internet-based crowdfunding, information cascades has a less detrimental effect, allowing better harnessing of the wisdom of the crowd.

Information efficiency is closely related to social welfare. In our model, for any strictly positive production cost $\nu \in (0, 1)$, it is socially costly to finance a type 0 project and socially beneficial to finance a type 1 project. As we discussed above, harnessing the wisdom from the crowd increases the information efficiency, resulting more efficient investment decision and thus improve the social welfare. Uni-directional cascade also means that offerings in the cascade model can fail whereas offerings never fail in the baseline model in Welch (1992). This would help us explain why some offerings fail occasionally and/or are withdrawn, without invoking insider information as Welch (1992) did in his model extension. By allowing some projects to fail when N is large (Proposition 2), we put the wisdom of the crowd to use to increase social welfare. To be specific, when N goes to ∞, the probability that a good project being financed goes to 1 while the probability that a bad project being implemented goes to 0.

4.3 proposer's Real Option and Information Aggregation

So far in our analysis we have required the proposer to implement the project according to the AoN threshold. In some cases in reality, especially when the entrepreneur also learns about the project's promise from crowdfunding (not knowing the true V in our model), he commits to AoN in fundraising, but still holds the real option on how to use the capital and information aggregated. For example, an entrepreneur successful on Kickstarter or Indigogo can still decide on the scale of the project and how much effort to put into developing the product. On some crowdfunding platforms, the entrepreneur can decide whether to use the capital raised explicitly or implicitly (by postponing product development indefinitely, which results in refunding the agents). Xu (2017) and Viotto da Cruz (2016) provide strong empirical evidence that the entrepreneur indeed use the information aggregated from crowdfunding platforms for real decisions.

Specifically, V can be interpreted as a transformation of the aggregate demand, which could be high ($V = 1$) or low ($V = 0$). Suppose that after the crowdfunding, an entrepreneur considers commercialization or abandoning the project (upon crowdfunding failure), and for simplicity the commercialization or continuation decision pays V (after normalization), but incurs an effort or reputation or monetary cost represented in reduced-form by I. Then the entrepreneur's expected payoff for the real option is

$$\max\{\mathbb{E}[V - I|\mathcal{H}_N], 0\} \tag{17}$$

recall $\mathcal{H}_N$ is the entire crowdfunding history, including information on the total number of supports out of N agents, and when an UP-cascade starts if there is one, etc. For a given pricing and AoN threshold, the final amount raised is directly informative on the quality of the project V:

Proposition 8. *The posterior belief on V is increasing in the equilibrium support observed. Conditional on failing to reach the AoN threshold, the proposer updates the belief more positively with more supporting agents.*

Even with a successful crowdfunding, the entrepreneur may still choose to forgo commercialization if his belief on V after crowdfunding is not sufficiently optimistic; likewise, despite crowdfunding failure, the entrepreneur may continue pursuing the project. Our model further predicts that the sensitivity of the update on V based on incremental supports is smaller conditional on fundraising success (reaching AoN threshold), because it likely involves an UP cascade and information aggregation is more limited.

Indeed, Xu (2017) documents in a survey of 262 unfunded Kickstarter entrepreneurs that after failing, 33% continued as planned. He also finds that a 50% increase in pledged amount leads to a 9% increase in the probability of commercialization outside the crowdfunding platform, which indicates a way smaller sensitivity. It would be interesting to understand how the entrepreneur designs AoN and pricing to not only maximize profit from the crowdfunding, but also increase the real option value, which constitutes interesting future work.

5 Discussion and Extensions

In this section, we characterize the equilibrium outcome when the agents have the options to wait, and when the number of agents is large but they do not necessarily play the subgame equilibrium specified earlier. The goal is two-fold: we want to demonstrate that our findings about the impact of AoN on financing and information aggregation efficiency are robust; we also want to demonstrate how the AoN feature leads to strategic considerations and equilibrium multiplicity that are absent in previous information cascade models, which are of theoretical interests.

5.1 Options to Wait

One common concern for standard information cascade models is the assumption of exogenous order of decision-making. In reality, agents may choose to wait in the hope that they may observe more information. Most results in standard information cascade models fail to hold if one introduces the option to wait. One particular feature of AoN is that the information aggregation pattern in our model is robust to the option to wait.

To be more specific, we enlarge each agent's action set to $\{S, R, W\}$, where W is the decision to wait and make decision after observing next agent's decision. The option to wait results in multiple equilibria due to the coordination problem on waiting decisions and off equilibrium path beliefs. That said, the following proposition shows that, in terms of information aggregation, there exists an equilibrium that is essentially the same as the one characterized in Proposition 3.

Proposition 9. *There exists an equilibrium such that:*

1. *Given the investment contribution (price) $c^* \in (0,1)$, the corresponding AoN threshold $T_N^* \leq N$ satisfies:*
$$\mathbb{E}(V|T_N^*, N) \leq c^* < \mathbb{E}(V|T_N^* + 1, N), \tag{18}$$
where $\mathbb{E}(V|x, N)$ is the posterior mean of V given there are x number of H signals out of N observations;

2. *Agents with signal H always support the project;*

3. *Agent i with signal L supports if there is already an UP cascade, that is:*

$$\mathbb{E}(V|K = k - 1) \geq c^*, \tag{19}$$

Otherwise, agent i with signal L chooses to wait until all agents has made a decision at least once. Let N_S be the number of agents that chooses to support as her first decision. Then agent i chooses to support if:

$$\mathbb{E}(V|N_S, N) \geq c^*, \tag{20}$$

and rejects otherwise.

Recall that with options to wait, k corresponds to the difference between the numbers of agents whose first time decision is support and agents whose first time decision is to wait. In terms of information aggregation, this equilibrium is equivalent to the one in Proposition 3: those agents who wait upon their first decision-making are exactly those who reject the project in the baseline model, and those who support upon their first decision-making are exactly those supporting agents in Proposition 3.

To see this, consider first if there is already an UP cascade then no one wants to deviate (if everyone chooses to invest once there is an UP cascade). Now if there is no cascade yet, then for agents with H signals, supporting always weakly dominates rejection and thus there is no need to wait. For agents with L signals, waiting till the end weakly dominates rejection and they will wait till the end. Observational learning still works since agents with different signals choose different actions. In the equilibrium, before the arrival of an UP cascade, all agents infer support action as a H signal and the decision to wait as a bad signal, resulting exactly the same information aggregation process as we describe in the baseline model.[11]

In terms of financing feasibility, this equilibrium is qualitatively the same as the one in Proposition 3. If $\nu > p$, the fundraising with $c > \nu$ would have a strictly positive success probability when the proposer commits to an AoN threshold. Thus our results on financing efficiency of good project is robust to options to wait, as summarized next.

[11] The option to wait may affect the optimal price c^*, because with the option to wait agents with L signal still contribute if the posterior after the information aggregation is good.

Proposition 10. *When N goes to infinity, the optimal price goes to* 1 *even when agents have the option to wait, and all good projects are financed for sure.*

We also remark that in crowdfunding, potential contributors often do not wait because of the high opportunity cost of attention. Moreover, the shares or products sold are often in limited supply, and waiting may cause an agent to miss out the opportunity.

5.2 Free-Rider Equilibria

Here we refer to the equilibrium discussed in Proposition 3 as "informer equilibrium" because before an UP cascade starts, every agent's action is informative. We first show that all other possible equilibria involve a group of "informers" and a group of "free-riders" whose actions before a cascade are ignored in equilibrium. We call this latter type of equilibrium "free-rider equilibrium".

Definition 1. *For an equilibrium strategy profile $\mathcal{A}(\cdot;\mathcal{H}_{i-1}) : \{L, H\} \to \{S, R\}$, we call agent i a "free-rider", if $A_i = S$ and $\mathbb{E}[V|\mathcal{H}_{i-1}] = \mathbb{E}[V|\mathcal{H}_i] < V_{k+1}$. In other words, agent i becomes free-rider following sub-history $\mathcal{H}_{i-1}$ if everyone knows that subsequent agents would not update their beliefs based on her action, even though an up-cascade has not been reached yet.*

Lemma 5. *An existing equilibrium is either informer equilibrium or free-rider equilibrium.*

In a free-rider equilibrium, given the history of actions, some agents may support regardless of her private signal even when the Up-cascade has not started yet. Moreover, who become free-riders is generally path-dependent. Those agents essentially delegate their investment decision to the gate-keeper agent and this is common knowledge. In other words, they free-ride on information aggregation from subsequent investors. Similar to the informer equilibrium, a free-rider equilibrium differs from the equilibria in most information-cascade models because coordination issues manifest themselves. Whether an agent becomes a free-rider depends on subsequent agents' beliefs and his beliefs on their beliefs, etc. Such phenomenon is absent in conventional models because the agent's expected payoff at the time of decision-making is independent of subsequent agents' actions; AoN breaks this independence and renders the supposedly sequential interaction similar to a simultaneous move game. The key difference between a free-rider and an agent weakly after an UP cascade starts is that the existence of free-rider relies on the fact that subsequent investors still take informative actions after the free-rider's move, and the information aggregation continues until a cascade starts or the game ends.

We also note that the existence of free-rider equilibria depends on the choice of (c, T). To see this, consider a free-rider equilibrium, if agent i supports regardless of her private bad signal before an UP cascade, then given $c \in (V_{k-1}, V_k]$, the following inequality must hold:

$$\varphi(V_k - c) + Q(V_{k-1} - c) \geq 0, \tag{21}$$

where φ is the probability that the Tth supporting agent is in an UP cascade and Q is the probability that the Tth supporting agent is not in an UP cascade, conditional on the history $\mathcal{H}_{i-1}$ and agent i's private L observation. Inequality (21) suggests that when agent i observes signal L, she has no incentive to reject. Since at agent i there is no UP-cascade yet, $Q > 0$, it must be the case $c < V_k$. Free-rider equilibrium cannot emerge if the proposer's payoff is dominated by that in the informer equilibrium when he sets $c \in \{V_k, K = -1, 0, \ldots N\}$. The existence of free-rider equilibria is not robust to option to wait. This is straightforward because for every agent observing signal L, she can be better off by waiting.

Free-rider equilibria can be viewed as derivatives of the equilibrium characterized in Proposition 3 in the sense that on each equilibrium path, if one excludes all free-riders, then sub-game dynamic is exactly the same as the one described in Proposition 1. The next proposition shows that in terms of information aggregation, free-rider equilibria deliver qualitatively the same result in the limit.

Proposition 11. *Let the number of informers in the sub-game equilibrium be $X_N(\hat{c}_N, T_N)$ when the proposer's endogenous design is $(\hat{c}_N, T_N)$, then for any positive integer l, as N goes to infinity, $Pr(X_N(\hat{c}_N, T_N) < l) \to 0$.*

The proposition implies that even in a free-rider equilibrium, the number of informers is unbounded as N goes up. This means for large N, the information aggregation efficiency relative to the standard information cascade settings goes up. Public information becomes arbitrarily informative as N goes to infinity.

Also building on the lemmas, we get

Proposition 12. *In any sequence of endogenous proposal designs* $\{\hat{c}_N, T_N\}_{N=1}^{\infty}$, $\hat{c}_N \to 1$ *as* N *goes to infinity.*

The proposition implies that as N becomes large, the proposer charges higher and higher price and still avoids DOWN cascade. This means no matter which equilibrium we select, in the limit the proposer can charge a high enough price to ensure that a good project is always financed. Our earlier finding that the financing efficiency improves is thus robust to equilibrium selection.

6 Conclusion

We incorporate AoN threshold into a classical model of information cascade, and find unidirectional cascades in which agents rationally ignore private signals and imitate preceding agents only if the preceding agents decide to support. Information production becomes more efficient, yielding more probable financing of good projects, and the weeding-outs of bad projects that is unattainable in earlier cascade models. In particular, when the number of agents grows large, equilibrium project implementation and information production approach the first best, even under information cascades.

As an important application of our model, financial technologies such as Internet-based funding platforms can help entrepreneurs reach out to a larger agent base, improve financing feasibility, and better harness the wisdom of the crowd, as envisioned by the regulatory authorities. We highlight that specific features and designs such as endogenous AoN thresholds are crucial in capitalizing on these potential benefits, especially with sequential sales and informational frictions. Further studies on platform designs taking into consideration the informational environment are definitely needed.

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Appendix: Derivations and Proofs

Proof of Lemma 1

Proof. Let k_n be the difference of numbers of H and L signals till the nth observation. For the prior, $k_0 = 0$, and $\frac{Pr(V=1)}{Pr(V=0)} = \frac{0.5}{0.5} = \frac{p^0}{q^0}$.

Suppose $\frac{Pr(V=1)|X}{Pr(V=0|X)} = \frac{p^{k_n}}{q^{k_n}}$ holds for $n \geq 0$, then for $n+1$:

1. If $X_{n+1} = H$, then $k_{n+1} = k_n + 1$, and

$$\begin{aligned}\frac{Pr(V=1|X)}{Pr(V=0|X)} &= \frac{\frac{Pr(X_{n+1}=H|V=1)Pr(V=1|X_1,X_2,\ldots,X_n)}{Pr(X_{n+1}=H)}}{\frac{Pr(X_{n+1}=H|V=0)Pr(V=0|X_1,X_2,\ldots,X_n)}{Pr(X_{n+1}=H)}} \\ &= \frac{Pr(X_{n+1}=H|V=1)p^{k_n}}{Pr(X_{n+1}=H|V=0)p^{k_n}} \\ &= \frac{p^{k_{n+1}}}{q^{k_{n+1}}};\end{aligned}$$

2. Similarly, if $X_{n+1} = L$, then $k_{n+1} = k_n - 1$, and

$$\begin{aligned}\frac{Pr(V=1|X)}{Pr(V=0|X)} =& \frac{\frac{Pr(X_{n+1}=L|V=1)Pr(V=1|X_1,X_2,\ldots,X_n)}{Pr(X_{n+1}=L)}}{\frac{Pr(X_{n+1}=L|V=0)Pr(V=0|X_1,X_2,\ldots,X_n)}{Pr(X_{n+1}=L)}} \\ =& \frac{Pr(X_{n+1}=L|V=1)p^{k_n}}{Pr(X_{n+1}=L|V=0)p^{k_n}} \\ =& \frac{p^{k_{n+1}}}{q^{k_{n+1}}};\end{aligned}$$

So $\frac{Pr(V=1|X)}{Pr(V=0|X)} = \frac{p^{k_{n+1}}}{q^{k_{n+1}}}$ holds for $n+1$. The lemma follows by induction. □

Proof of Proposition 1

Proof. Consider a sequence of action $\mathcal{H} \in \{0,1\}^N$. Denote $\mathcal{H}_i$ as the subsequence of the first i actions in $\mathcal{H}$. $N_S(\mathcal{H}_i)$ shows the number of supporting agents in $\mathcal{H}_i$, and note that $2N_S(\mathcal{H}_i) - i$, is the difference between supporting and rejecting agents in $\mathcal{H}_i$. Therefore, the proposal is accepted if $N_S(\mathcal{H} \geq T_N$. In this case, we call agent g the "gate-keeper" if g is the smallest number for which $N_S(\mathcal{H}_g) = T_N$.

Lemma 6. *Suppose $V_{k-1} < c \leq V_k$ for some $k > 0$. Then, in every equilibrium, the following relation holds for every $2 \leq i \leq N$:*

$$\mathbb{E}[V|\mathcal{H}_{i-1}] \leq V_{k+1} \tag{22}$$

In other words, there is an upper-bound on the expected value of the project that is dependent on the *supporting cost/contribution.*

Proof. Suppose the contrary and the expected value exceeds V_{k+1}. Then there exists an agent u such that $\mathbb{E}[V|\mathcal{H}_{u-1}] = V_{k+1}$. Note that u accepts the proposal regardless of her private signal because

$$\mathbb{E}[V|\mathcal{H}_{u-1}] \geq \mathbb{E}[V|\mathcal{H}_{u-1}, X_u = L] = V_k \geq c$$

when the project is implemented, and she pays nothing if the AoN threshold is not reached. In other words, an UP cascade starts and u's action is not informative for the subsequent agents. $\mathbb{E}[V|\mathcal{H}_i] = V_{k+1}$, for every $i \geq u-1$. It contradicts the assumption that the expected value exceeds V_{k+1} for some agent. This proves the lemma. □

First, it is obvious that once an UP cascade starts, every subsequent agent supports the proposal, no matter how off-equilibrium deviation is perceived. To see this, note that if $\mathbb{E}[V|\mathcal{H}_{i-1}] = V_{k+1}$, then

$$\mathbb{E}[V|X_i, \mathcal{H}_{i-1}] \geq \mathbb{E}[V|X_i = L, \mathcal{H}_{i-1}] = V_k \geq c \Rightarrow \mathbb{E}[V|\mathcal{H}_{i-1}] = \mathbb{E}[V|\mathcal{H}_i]$$

Now Suppose an UP cascade has not started yet and $N_S(\mathcal{H}_{i-1}) < T_N - 1$, i.e. before agent i's action, the AoN threshold has not been reached yet. If $X_i = H$ and she rejects the proposal, with a positive probability the AoN threshold is reached without her support. In this case, agent i would lose from her deviation because

$$\mathbb{E}[V|X_i = H, N_S(\mathcal{H}) \geq T_N] \geq \mathbb{E}[V|N_S(\mathcal{H}) \geq T_N] = \mathbb{E}[\mathbb{E}[V|\mathcal{H}_g]|N_S(\mathcal{H}) \geq T_N] \geq c$$

If $X_i = L$ and she supports the project, the AoN threshold is reached with a positive probability. Also note that $\mathbb{E}[V|\mathcal{H}] \leq V_{k+1}$ in any equilibrium by Lemma 6. Finally, according to the equilibrium specification, every agent before reaching an UP cascade or AoN threshold plays based on her private signal. Therefore

$$\begin{aligned} &\mathbb{E}[V|X_i = H, \mathcal{H}] \leq V_{k+1} \quad \forall \ \mathcal{H} \in \{0,1\}^N \\ \Rightarrow &\mathbb{E}[V|X_i = L, \mathcal{H}] \leq V_{k-1} < c \end{aligned} \tag{23}$$

We thus have $\mathbb{E}[V|X_i = L, \mathcal{H}] < c$ for any history of events, including the ones that the AoN threshold is reached. Again, the deviation is costly. We have shown that there is a Bayesian Nash equilibrium such that before reaching an UP-cascade or the AoN threshold, each agent accepts if and only if she receives a high signal, i.e., agent i plays according to her private signal if $\mathbb{E}[V|H_u] \leq V_k$ for every $1 \leq u < i$ and $N_S(\mathcal{H}_i) < T_N - 1$.

Finally, when $N_S(\mathcal{H}_{i-1}) \geq T_N - 1$, the agent's support leads to investment for sure. The agent thus accepts the proposal if and only if $\mathbb{E}[V|\mathcal{H}_{i-1}, X_i] \geq c$. Therefore, the strategy profile provided in Proposition 1 is an equilibrium.

To show the informer equilibrium characterized in Proposition 1 is unique, notice that agents with H signals finds supporting a strictly dominating strategy whenever it is still possible to reach the AoN threshold. We therefore only focus on agents with L signals. In (23), we used the fact that a low-signal agent misleads the subsequent agents by accepting the proposal, thus she cannot benefit from supporting. In other words, since the beliefs do not improve once an UP cascade starts, she would always lose from manipulating the subsequent agents to start the UP cascade too early. □

Proof of Proposition 2

Proof. Because when $V = 1$, $Pr(X = H|V = 1) = p > q$, it is known that (Feller (1968), page 347 equation 2.8):

$$Pr(C_o) = 0$$

where C_o indicates the event that there is no UP cascade. When there is a cascade, then as $N \to \infty$, $\frac{N_S}{N} \to 1$, so the AoN threshold would be reached for sure and the project is implemented. □

Proof of Lemma 2

Proof. For agent 1, her posterior after observing H is $\mathbb{E}(V|X_1 = H) = p$. If $c > p$, then agent 1 chooses rejection regardless of her private signal and a DOWN cascade starts from the beginning for sure.

Since $c = 1 - p = \mathbb{E}(V|K = -1)$ will induce an UP cascade starting form the beginning for sure, the entrepreneur has no incentive to charge $c < 1 - p$. Combine with $c \leq p$ we have $c \in [1 - p, p]$. Also, for each $c \in (V_{k-1}, V_k]$, $c = V_k$ induces exactly the same number of supporting agents, so in the equilibrium proposer always charges $c^* = V_k$ for some $k \in \{-1, 0, 1, \ldots, N\}$. Without loss of generality, only three prices are possible: $c_{-1} = 1 - p$, $c_0 = \frac{1}{2}$ and $c_1 = p$. Let $\Pi(c, N)$ be the expected profit when the price is c and there are $N \geq 2$ potential agents. It is clear that $\Pi(c, N)$ is increasing in N.

$c = 1 - p$: The total profit is $(1 - p)N$;

$c = \frac{1}{2}$: After the first two observations, LL induces a DOWN cascade, HL and HH both induce an UP cascade at agent 1, and LH does not change the belief. The expected profit is $\Pi(c, N) = \frac{p+q}{2}\frac{1}{2}N + \frac{qp+pq}{2}(\frac{1}{2} + \Pi(c, N-2)) \leq \frac{1}{4}N + pq(\frac{1}{2} + \Pi(c, N))$. Thus $c = \frac{1}{2}$ is dominated by $c = 1 - p$ if:

$$\Pi(c, N) \leq \frac{\frac{N}{4} + \frac{pq}{2}}{1 - pq} \leq (1 - p)N \ for \ N = 2, 3, \ldots \tag{24}$$

One can verify that the inequality holds for $p \in (\frac{1}{2}, \frac{3}{4} + \frac{1}{4}(3^{\frac{1}{3}} - 3^{\frac{2}{3}})]$;

$c = p$: After the first two observations, HH induces an UP cascade, LL and LH both induce a DOWN cascade at agent 1, and LH does not change the belief. The expected profit is $\Pi(c, N) = \frac{p^2+q^2}{2}pN + \frac{qp+pq}{2}(p+$

$\Pi(c, N-2)) \leq \frac{p^2+q^2}{2}pN + pq(p + \Pi(c, N))$. Thus $c = p$ is dominated by $c = 1-p$ if:

$$\Pi(c, N) \leq \frac{\frac{p^2+q^2}{2}pN + p^2q}{1-pq} \leq (1-p)N \ for \ N = 2, 3, \dots \tag{25}$$

One can verify that the inequality holds for $p \in (\frac{1}{2}, \frac{3}{4} + \frac{1}{4}(3^{\frac{1}{3}} - 3^{\frac{2}{3}})].0$ □

Proof of Proposition 3

Proof. We have shown in Proposition 1 that the agents' strategy profile in the current proposition constitutes a subgame perfect Nash equilibrium for every (c, T), we therefore only need to prove that (c^*, T^*) are indeed the optimal price and AoN threshold, respectively.

Step 1: Optimal T_N when $c > \frac{1}{2}$

Suppose k is the smallest integer such that $V_k = \mathbb{E}[V|K = k] \geq c$. Then, define $T_N^* = \lfloor \frac{k+N}{2} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x. We show T_N^* is the optimal AoN threshold.

The proposer's problem is to choose T_N to maximize the expected number of supporters, given the equilibrium specified in Proposition 1. Denote $P(T_N)$ and $S(T_N)$ as the probability of the project implementation and its expected number of supporters, respectively. Therefore, the objective function of the proposer is to maximize $\pi(T_N) = P(T_N)S(T_N)$. To prove the proposition, we first show that $\pi(T_N) \leq \pi(T_N^*(c^*))$, for every $T_N > T_N^*(c^*)$. Then, we show both $S(.)$ and $T(.)$ are strictly increasing for $T_N < T_N^*(c^*)$.

Given that we work with the sequence of signals $\mathcal{S} \in \{H, L\}^N$. The following definition is useful for the analysis:

*A sequence of signals $\mathcal{S} \in \{H, L\}^N$ is T_N**-supported** if the AoN threshold is reached when the sequence of signals is $\mathcal{S}$ and the proposer sets the acceptance requirement T_N.*

For the first part of the argument, we have the following result.

Lemma 7. *Consider $T_N^*(m) < T_N \leq N$.*

***(a)** For $T_N > T_N^*(m)$, if sequence $\mathcal{S}$ is T_N-supported then it is also T_N^*-supported.*

***(b)** For $T_N > T_N^*(m)$, there exists a sequence $\mathcal{S}$ such that it is T_N^*-supported but not T_N-supported.*

Proof. **Part (a)**

Note first that if $k + N$ is even, then for any $T_N > \lceil \frac{k+N}{2} \rceil$, every T_N-supported sequence is $\lceil \frac{k+N}{2} \rceil$-supported as well. The reason is that by the $\lceil \frac{k+N}{2} \rceil$'th supporting agent an UP cascade starts. Therefore, decreasing the AoN threshold to $\lceil \frac{k+N}{2} \rceil$ does not eliminate any supported sequence.

Next, we show that if $k + N$ is odd, every T_N-supported sequence is $\lfloor \frac{k+N}{2} \rfloor$-supported sequence. To see this, suppose $k + N$ is odd and $\mathcal{S}$ is a T_N-supported sequence and is not $\lfloor \frac{k+N}{2} \rfloor$-supported. Suppose $h(\lfloor \frac{k+N}{2} \rfloor - 1)$ is the agent that receives the $\left(\lfloor \frac{k+N}{2} \rfloor - 1\right)$'th H signal. Note that by changing the requirement from T_N to $\lfloor \frac{k+N}{2} \rfloor$, for $i \leq h(\lfloor \frac{k+N}{2} \rfloor - 1)$, the agents' strategy does not change. Therefore, if there is an UP cascade, then it is the same as for the $\lfloor \frac{k+N}{2} \rfloor$ case. If no UP cascade starts by the $h\left(\lfloor \frac{k+N}{2} \rfloor - 1\right)$'th agent, lest it contradicts the assumption that $\mathcal{S}$ is not $\lfloor \frac{k+N}{2} \rfloor$-supported. As a result,

$$V_k \geq \mathbb{E}\left[V \mid \lfloor \frac{k+N}{2} \rfloor - 1 \text{ out of } h(\lfloor \frac{k+N}{2} \rfloor - 1)\right]$$
$$\Rightarrow k \geq \lfloor \frac{k+N}{2} \rfloor - 1 - \left(h\left(\lfloor \frac{k+N}{2} \rfloor - 1\right) - \lfloor \frac{k+N}{2} \rfloor + 1\right)$$
$$\Rightarrow h\left(\lfloor \frac{k+N}{2} \rfloor - 1\right) \geq N - 2$$

Therefore, at most 2 agents are in the line after $h(\lfloor \frac{k+N}{2} \rfloor - 1)$. Since $\mathcal{S}$ is T_N-supported and $T_N > \lceil \frac{k+N}{2} \rceil$, then both following agents support the proposal, and it is $\lfloor \frac{k+N}{2} \rfloor$-supported. This is a contradiction and part (a) must be true.

Proof of Part (b)

For even $k+N$, consider the sequence $\mathcal{S}^1 = (\underbrace{L, \cdots, L}_{\frac{N-k}{2}}, \underbrace{H, \cdots, H}_{\frac{N+k}{2}})$. It is $\frac{N+k}{2}$ -supported sequence because:

$$\mathbb{E}\left[V \mid \frac{N+k}{2} \text{ out of } N-2\right] = V_k \geq c$$

It is easy to see that it is not T_N-supported for $T_N > \frac{N+k}{2}$.

For odd $N+k$, consider the sequence $\mathcal{S}^2 = (\underbrace{L, \cdots, L}_{\frac{N-k-1}{2}}, \underbrace{H, \cdots, H}_{\frac{N+k-1}{2}}, L)$. It is $\lfloor \frac{k+N}{2} \rfloor$ -supported because:

$$\mathbb{E}\left[V \mid \lfloor \frac{k+N}{2} \rfloor \text{ out of } N-1\right] = V_k \geq c$$

It is clear that $\mathcal{S}^2$ cannot be T_N-supported for $T_N > \lfloor \frac{k+N}{2} \rfloor$. □

Now, we show that for every $1 < T_N \leq T_N^* = \lfloor \frac{k+N}{2} \rfloor$, $\pi(T_N) > \pi(T_{N-1})$. But consider the following useful lemma first.

Lemma 8. *If sequence $\mathcal{S}$ is $T_N - 1$-supported and not T_N-supported, then at least two consecutive L signals follow the $T_N - 1$'th H signal in $\mathcal{S}$.*

Proof. Let g be $(T_N - 1)$th supporting agent. First, we show that $g \leq N-2$. To see this, note that all rejecting agents before agent g should have received a low signal. Therefore,

$$V_k \leq \mathbb{E}[V \mid \mathcal{H}_{g-1}, X_g] \leq \mathbb{E}[V \mid T_N - 1 \text{ out of } g] = \mathbb{E}[V \mid T_N \text{ out of } g+2]$$
$$\leq \mathbb{E}\left[V \mid \lfloor \frac{k+N}{2} \rfloor \text{ out of } g+2\right] \Rightarrow g+2 \leq N$$

The last inequality derives from the definition of T_N^*. Because $\mathcal{S}$ is not T_N-supported, none of agents $g+1$ and $g+2$ supports the proposal. Therefore, they should receive a low signal; otherwise, at least one of them should have supported the proposal. $\square$

Now, we show $P(T_N) \geq P(T_N - 1)$, for $T_N \leq T_N^*$. For every sequence like $\mathcal{S} = (\dots, H, L, L, \dots)$ that is $T_N - 1$-supported and is not T_N-supported, consider the sequence $\mathcal{S}' = (\dots, L, H, H, \dots)$, in which only the three middle signals are reversed. It is easy to see that $\mathcal{S}'$ is not $T_N - 1$-supported, while it is T_N-supported. Moreover:

$$Pr(\mathcal{S}') = Pr(\mathcal{S}'_{g+2}) = \frac{1}{2}\left[Pr(\mathcal{S}'_{g+2}|V=1) + Pr(\mathcal{S}'_{g+2}|V=0)\right] =$$
$$\frac{1}{2}\left[\frac{p}{1-p}Pr(\mathcal{S}_{g+2}|V=1) + \frac{1-p}{p}Pr(\mathcal{S}_{g+2}|V=0)\right]$$
$$\Rightarrow Pr(\mathcal{S}') - Pr(S) = \frac{2p-1}{p}\left[\frac{1}{1-p}Pr(S_{g+2}|V=1) - \frac{1}{p}Pr(S_{g+2}|V=0)\right]$$

where, $\mathcal{S}_{g+2}$ is the subsequence of the first $g+2$ signals in $\mathcal{S}$. Consequently, we only need to show $\frac{Pr(S_{g+2}|V=1)}{Pr(S_{g+2}|V=0)} \geq \frac{1-p}{p}$. To see this, note that

$$\mathbb{E}[V|\mathcal{S}_g] = V_k \Rightarrow \mathbb{E}[V|\mathcal{S}_{g+2}] = V_{k-2} \Rightarrow \frac{Pr(S_{g+2}|V=1)}{Pr(S_{g+2}|V=0)} = \frac{V_{k-2}}{1-V_{k-2}} \geq \frac{c_{-1}}{1-c_{-1}} = \frac{1-p}{p} \quad \forall k > 0$$

We thus have $P(T_N) > P(T_N - 1)$ for $T_N \leq T_N^*$ and $k > 0$.

Furthermore, notice that the number of supporting agents in any T_N-supported sequence exceeds the number of supporting agents in any $T_N - 1$-supported sequence that is not T_N-supported. As a result, $S(T_N) > S(T_N - 1)$, which leads to $\pi(T_N) > \pi(T_N - 1)$ for $2 \leq T_N \leq T_N^*$.

Step 2: Optimal T_N when $c \leq \frac{1}{2}$

The proof of $\pi(T_N^*) \geq \pi(T_N)$ for $T_N > T_N^*$ is similar to the case of $c > \frac{1}{2}$.

For the other case, we separately consider two scenarios: $c \leq c_{-1} = 1-p$ and $c \in (c_{-1}, \frac{1}{2}]$. Since any $c \in (V_{k-1}, V_k]$ results in the same investment decisions, without loss of generality, we focus on cases $c \in \{V_k\}, k = 0, -1, \dots$.

Consider first the scenario that $c \leq c_{-1}$, that is to say, $k \leq -1$. The UP cascade starts from the first agent for sure, so any AoN threshold $T_N^* \leq N$ is optimal.

Consider next the scenario that $c = \frac{1}{2}$, that is to say, $k = 0$. Similar to the $c^* > \frac{1}{2}$ case, T_N threshold only dominates $T_N + 1$ threshold along the HLL path. Let $Q_{T_N^*}$ be the event that there is no UP cascade yet and at the T_Nth supporting agent there are exactly equal numbers of supporting and rejecting agents (that is to say, the T_Nth supporting agent is the $2T_N$th agent). Let U_{2T_N+1} be the event that the UP cascade arrives at the $2T_N + 1$th agent. Event U_{2T_N+1} happens if and only if Q_{T_N} happens and the $2T_N + 1$th agent observes a H signal, because by this point AoN threshold is already met and we are back to the standard cascade setting. Based on Lemma 4 for the case of $k = 1$, we have:

$$\begin{aligned} P(Q_{T_N}) =& \frac{1}{2p} P(U_{2T_N+1}|V=1) + \frac{1}{2q} P(U_{2T_N+1}|V=0) \\ =& \frac{1}{2T_N+1} \begin{pmatrix} 2T_N+1 \\ T_N+1 \end{pmatrix} (pq)^{T_N}, \end{aligned}$$

and the expected profit from HLL path (implementable with T_N but not with $T_N + 1$) is:

$$E_{HLL} \equiv (c^* - \nu) T_N P(Q_{T_N}) P(LL \text{ } for \text{ } 2T_N + 1 \text{ } and \text{ } 2T_N + 2).$$

Similarly, let Q_{T_N+1} be the event that there is no UP cascade yet and at the $T_N + 1$th supporting agent there are exactly equal numbers of supporting and rejecting agents (that is to say, the $(T_N + 1)$th supporting agent is the $(2T_N + 2)$th agent). When the threshold is $T_N + 1$, the probability that the project would be implemented with the $T_N + 1$ threshold but fails in the T_N threshold (since the T_Nth H signal agent behave differently given different AoN threshold) is:

$$P_1 \equiv P(Q_{T_N+1}) - P(Q_{T_N}) pq,$$

where the second term is the case that the event Q_{T_N} realizes and agent $i + 1$ and $i + 2$ observe L and H, respectively. Note that Q_{T_N+1} indicates the $T_N + 1$th supporter sees equal number of supporting and rejection actions (including her own), thus HLH meetings both funding threshold $T_N + 1$ and T_N with the same payoff to the proposer.

The ratio of the expected profit from HLL path that meets T_N but not $T_N + 1$, to that from paths implemented with $T_N + 1$ threshold but not T_N is:

$$\frac{E_{HLL}}{(c^* - \nu)(T_N + 1)P_1} = \frac{(p^2 + q^2)(T_N + 2)}{6pq(T_N + 1)} \leq \frac{p^2 + q^2}{4pq},$$

where the last inequality comes from the fact that $T_N \geq 1$. Since $p^2 + q^2 + 2pq = (p + q)^2 = 1$, $\frac{p^2+q^2}{4pq} < 1$ is equivalent to $pq > \frac{1}{6}$. So when $pq > \frac{1}{6}$, T_N is strictly dominated by $T_N + 1$.

When $pq \leq \frac{1}{6}$, we have $p \geq \frac{1}{2} + \frac{\sqrt{3}}{6} > \frac{3}{4}$. We now show that strategy T_N and $c^* = \frac{1}{2}$ is strictly dominated by alternative strategy $c^* = p$ (so $k = 1$) and AoN threshold $T_N + 1$. For $c^* = \frac{1}{2}$ and AoN threshold T_N, we have shown earlier that the project would be implemented either there is an UP cascade before/at agent $2T_N - 1$ or there is no UP cascade before $2T_N$ but the $2T_N$th agent is the T_Nth supporting agent. It suffices to show that in either scenario, the alternative strategy fares better for the proposer.

1. When there is an UP cascade before $2T_N$, consider the case that right after the cascade the next agent

observes H signal and support. This would also result in an UP cascade for $(c^* = p, T_N + 1)$ and the same number of supporting agents. The conditional probability that the next agent observes H is $\mathbb{E}(V = 1|K = 1)p + \mathbb{E}(V = 0|K = 1)q = p^2 + q^2 = 1 - 2pq \geq \frac{2}{3}$. For the case $(c^* = p, T_N + 1)$, for each contribution the proposer charges p instead of $\frac{1}{2}$. The proposer receives higher expected payoffs from UP cascades because $p(p^2 + q^2) > \frac{3}{4}(p^2 + q^2) \geq \frac{1}{2}$.

2. When there is no UP cascade before $2T_N$ but the $2T_N$th agent is the T_Nth supporting agent (event Q_{T_N}), consider two corresponding scenarios in $(c^* = p, T_N + 1)$: (i) Event Q_{T_N} happens and the next agent observes H and support; (ii) There is no UP cascade (corresponding to $c^* = p$, that is to say, $k + 1 = 2$) yet, but there is one more supporting agent by (and including) the $2T_N - 1$th agent, and the $2T_N$th and $2T_N + 1$th agents observe L and H, respectively.

   In both cases, funding threshold $T_N + 1$ is met and there are at least the same number of supporting agents as in Q_{T_N}. For (i), conditional on there are equal number of supporting and rejecting agents at $2T_N$, the conditional probability that the next agent observes H is $\mathbb{E}(V = 1|K = 0)p + \mathbb{E}(V = 0|K = 0)q = \frac{1}{2}$. For (ii), similar to the discussion on $P(Q_{T_N})$, the probability of scenario (ii) is:

$$\frac{1}{2T_N}\begin{pmatrix} 2T_N \\ \frac{2T_N+2}{2} \end{pmatrix}(pq)^{T_N} = \frac{1}{2}P(Q_{T_N}).$$

   The probability that either (i) or (ii) happens equals $P(Q_{T_N})$, and in either case there are at least the same number of supporting agents paying $p > \frac{1}{2}$. So for $(c^* = p, T_N + 1)$ the proposer receives more payoffs when there is no UP cascade before $2T_N$. Thus the proposer is strictly better off with strategy $(c^* = p, T_N + 1)$.

In conclusion, $T_N = T_N(c^*)$ is the proposer's weakly dominating strategy, and it is a strictly dominating strategy whenever different T_N choices may lead to different equilibrium outcomes.

Step 3: The Existence of Optimal c

First of all, if $c \leq V_{-1}$, then an UP cascade would start from the beginning and the total return is $(c - \nu)N$. If $c > V_N$, then no agent would invest and the total return is 0. Thus we focus on the case $c \in [V_{-1}, V_N]$.

For any $c \in (V_{k-1}, V_k]$, $k \in \{0, 1 \dots N\}$, in the subgame the agents follow the same equilibrium strategy profile specified in Proposition 1. Thus for the proposer, any $c \in (V_{k-1}, V_k)$, $k \in \{0, 1 \dots N\}$ is dominated by $c = V_k$. Consequently, $c^* \in \{V_k | k = -1, 0 \dots N\}$. The existence of optimal c^* comes from the fact that the set $\{V_k | k = -1, 0 \dots N\}$ is finite. □

Proof of Lemma 4

Proof. Since an UP cascade starts once there are k more H signals. Exactly k more H signals at agent i implies $\frac{i-k}{2}$ L signals and $\frac{i+k}{2}$ H signals. The number of L signals suggests that $i \geq K$, and the number of H signals implies that $i+k$ must be even. There are $C_i^{\frac{i+k}{2}}$ different potential paths to reach exactly k more H signals, and for each path, the possibility is $p^{\frac{i+k}{2}} q^{\frac{i-k}{2}}$ conditional on $V=1$ and $q^{\frac{i+k}{2}} p^{\frac{i-k}{2}}$ conditional on $V=0$. Then:

$$Pr(\text{exactly } k \text{ more } H \text{ signals at agent } i) = \begin{pmatrix} i \\ \frac{i+k}{2} \end{pmatrix} (pq)^{\frac{i-k}{2}} \frac{p^k + q^k}{2}$$

By the reflection principle and Lemma 3 one can infer that $\varphi_{k,i} = \frac{k}{i} Pr(\text{exactly } k \text{ more } H \text{ signals at agent } i)$. That is:

$$\varphi_{k,i} = \frac{k}{i} \begin{pmatrix} i \\ \frac{i+k}{2} \end{pmatrix} (pq)^{\frac{i-k}{2}} \frac{p^k + q^k}{2}.$$

□

Proof of Proposition 4

Proof. For $c = c_{-1} = 1-p$, the project will be financed for sure. For $c = V_{k-1}$ $k \in \{1, 2, \ldots, N\}$, an UP cascade starts once there are k more supporting agents. When an UP cascade occurs at agent i, all subsequent agents support the project and the financing is successful, there would be in total $N - \frac{i-k}{2}$ supporting agents, and each contributes $c = V_{k-1}$. An UP cascade occurs only when $i+k$ is even. If $N+k$ is odd and there is no UP cascade yet, then the project may still reach the AoN threshold if there are exactly $k-1$ more supporting agents at agent N. Suppose there is one more round $N+1$, then an UP cascade starts at agent $N+1$ if and only if there are exactly $k-1$ more supporting agents at agent N and agent $N+1$ observes H. That is to say, when $k+N$ is odd, the probability that there is no UP cascade and the project reaches the AoN threshold is $\frac{p^k+q^k}{p^{k+1}+q^{k+1}} \varphi_{k,N+1}$, and there would be $\frac{N+k-1}{2}$ supporting agents in total. Similarly, if $N+k$ is even and there is no UP cascade until agent $N-1$ yet, then the project may still reach the AoN threshold if there are exactly $k-1$ more supporting agents at agent $N-1$. The event can be further decomposed into two parts. The first event is that the UP cascade starts at agent N, and the corresponding probability is $\frac{p^{k-1}+q^{k-1}}{p^k+q^k} \varphi_{k,N}$, and there would be $\frac{N+k}{2}$ supporting agents in total. The second event is that there is no UP cascade and there are exactly T_N^* supporting agents at agent $N-1$ (so the last agent observes L and rejects), and the corresponding probability is $\frac{qp^{k-1}+q^{k-1}p}{p^k+q^k} \varphi_{k,N}$, and there would be $\frac{N+k-2}{2}$ supporting agents in total.

To show the existence of $\underline{N}(k)$, we first prove the existence of $\underline{N}(0)$, then proceed to the $k \geq 1$ case. $\pi(c_{-1}, N) = (1-p-\nu)N$. When $c = c_0 = \frac{1}{2}$, an UP cascade starts once there are more than 1 H signals. From standard Gambler's ruin problem we know that the conditional probability that an UP cascade occurs at sometime is 1 if $V=1$, and $\frac{q}{p}$ if $V=0$ (Feller (1968), page 347 equation 2.8). Because $pq = p(1-p) < \frac{1}{4}$, we have:

$$\begin{aligned}
(c_0-\nu)(Pr(V=1)+Pr(V=0)\frac{q}{p}) &= (\frac{1}{2}-\nu)(\frac{1}{2}+\frac{1+\frac{1-p}{p}}{2}) \\
&= (\frac{1}{2}-\nu)\frac{1}{2p} \\
&> 1-p-\nu \\
&= c_{-1}.
\end{aligned}$$

Since $\varphi_{0,i}$ is strictly positive, there exists a strictly positive integer $N_1(0)$ such that:

$$(c_0-\nu)\sum_{i=1}^{N_1(0)}\varphi_{0,i} > 1-p-\nu.$$

Let $D = (c_0-\nu)\sum_{i=1}^{N_1(0)}\varphi_{0,i} - (1-p-\nu) > 0$, $Q = (c_0-\nu)\sum_{i=1}^{N_1(0)}\varphi_{0,i}\frac{i}{2}$, and $\underline{N}(0)$ be the smallest integer that is larger than $\max\{N_1(0), \frac{Q}{D}\}$. Then for any $N \geq \underline{N}(0)$:

$$\begin{aligned}
\pi(c_0, N) &\geq (c_0-\nu)\sum_{i=1}^{\underline{N}(0)}\varphi_{0,i}(N-\frac{i}{2}) \\
&= N(c_0-\nu)\sum_{i=1}^{\underline{N}(0)}\varphi_{0,i} - Q \\
&\geq \frac{Q}{D}D + (1-p-\nu)N - Q \\
&= (1-p-\nu)N.
\end{aligned}$$

Now consider the case $k \geq 1$ (when $\nu \leq V_k$). When the price is V_{k-1}, an UP cascade starts once there are more than k H signals. It occurs once there are more than $k+1$ H signals when the price is V_k. For both cases, the conditional probability that an UP cascade occurs at sometime is 1 if $V=1$. When $V=0$, the conditional probability that an UP cascade occurs at sometime is $\frac{q^k}{p^k}$ for V_{k-1} and $\frac{q^{k+1}}{p^{k+1}}$ for V_k, respectively (Feller (1968), page 347 equation 2.8).

For each $k \geq 1$, and the time i arrival rate $\varphi_{k,i}$, there exists a corresponding $\varphi_{k+1,i+1}$ for price V_k. For each i, we have:

$$\begin{aligned}
\frac{(V_k-\nu)\varphi_{k+1,i+1}}{(V_{k-1}-\nu)\varphi_{k,i}} \geq \frac{V_k\varphi_{k+1,i+1}}{V_{k-1}\varphi_{k,i}} &= \frac{V_k\frac{k+1}{i+1}\frac{(i+1)!}{\frac{i+k+2}{2}!\frac{i-k}{2}!}(pq)^{\frac{i-k}{2}}\frac{p^{k+1}+q^{k+1}}{2}}{V_{k-1}\frac{k}{i}\frac{i!}{\frac{i+k}{2}!\frac{i-k}{2}!}(pq)^{\frac{i-k}{2}}\frac{p^k+q^k}{2}} \\
&= p\frac{k+1}{k}\frac{i}{\frac{i+k}{2}+1}(1+\frac{(pq)^{k-1}(p-q)^2}{(p^k+q^k)^2}).
\end{aligned}$$

Since $\lim_{i\uparrow\infty} p\frac{i}{\frac{i+k}{2}+1} = 2p > 1$, for each k, the ratio $\frac{V_k\varphi_{k+1,i+1}}{V_{k-1}\varphi_{k,i}}$ is monotonically increasing in i and there exists an integer N_1 that $\frac{V_k\varphi_{k+1,i+1}}{V_{k-1}\varphi_{k,i}} \geq 1$ whenever $i \geq N_1$.

Because

$$\begin{aligned}(p^{k+1}+q^{k+1})(p^{k-1}+q^{k-1}) &= p^{2k}+q^{2k}+p^{k+1}q^{k-1}+p^{k-1}q^{k+1} \\ &= p^{2k}+q^{2k}+p^{k-1}q^{k-1}(p^2+q^2) \\ &> p^{2k}+q^{2k}+p^{k-1}q^{k-1}(2pq) \\ &= (p^k+q^k)^2.\end{aligned}$$

We have

$$\begin{aligned}\lim_{N\uparrow\infty}(V_k-\nu)\sum_{i=1}^{N-1}\varphi_{k+1,i+1} &= (V_k-\nu)\left(\frac{1}{2}+\frac{\frac{q^{k+1}}{p^{k+1}}}{2}\right) \\ &= \frac{V_k-\nu}{V_k}\frac{1}{2}\frac{p^k}{p^k+q^k}\frac{p^{k+1}+q^{k+1}}{p^{k+1}} \\ &= \frac{V_k-\nu}{V_k}\frac{1}{2p}\frac{p^{k+1}+q^{k+1}}{p^k+q^k} \\ &> \frac{V_k-\nu}{V_k}\frac{1}{2p}\frac{p^k+q^k}{p^{k-1}+q^{k-1}} \\ &= \frac{V_k-\nu}{V_k}V_{k-1}\left(\frac{1}{2}+\frac{\frac{q^k}{p^k}}{2}\right) \\ &\geq (V_{k-1}-\nu)\left(\frac{1}{2}+\frac{\frac{q^k}{p^k}}{2}\right) \\ &= \lim_{N\uparrow\infty}V_{k-1}\sum_{i=1}^{N}\varphi_{k,i}.\end{aligned}$$

Given $\lim_{N\uparrow\infty}(V_k-\nu)\varphi_{k+1,i+1}\downarrow 0$, there exists an integer $N_2 \geq N_1$ such that:

$$D \equiv (V_k-\nu)\sum_{i=1}^{N_2-1}\varphi_{k+1,i+1} - (V_{k-1}-\nu)\sum_{i=1}^{N_2}\varphi_{k,i} - (V_{k-1}-\nu)\frac{pq^{k-1}}{p^k+q^k}\varphi_{k,N_2} - (V_{k-1}-\nu)\frac{p^k+q^k}{p^{k+1}+q^{k+1}}\varphi_{k,N_2+1} > 0$$

Let $Q \equiv (V_{k-1}-\nu)\sum_{i=1}^{N_2}\varphi_{k,i}\frac{i-k}{2} - (V_k-\nu)\sum_{i=1}^{N_2-1}\varphi_{k+1,i+1}\frac{i-k}{2}$. Then for each k, let $\underline{N}(k)$ be the smallest integer that is larger than $\max\{N_2, \frac{Q}{D}\}$. Then for any $N \geq \underline{N}(0)$:

$$\begin{aligned}\pi(V_k, N) - \pi(V_{k-1}, N) >&\pi(V_k, \underline{N}(k)) - \pi(V_{k-1}, \underline{N}(k)) \\ >&\underline{N}(k)(V_k - \nu) \sum_{i=1}^{N_2-1} \varphi_{k+1,i+1} - (V_{k-1} - \nu)\tfrac{p^k+q^k}{p^{k+1}+q^{k+1}}\varphi_{k,\underline{N}(k)+1}\tfrac{\underline{N}(k)+k-1}{2} \\ &- (V_{k-1} - \nu)\sum_{i=1}^{N_2} \varphi_{k,i} - Q - (V_{k-1} - \nu)\tfrac{p^{k-1}q+pq^{k-1}}{p^k+q^k}\varphi_{k,\underline{N}(k)}\tfrac{\underline{N}(k)+k-2}{2} \\ >&\underline{N}(k)D - Q \geq \frac{Q}{D}D - Q = 0.\end{aligned}$$

□

Proof of Corollary 3

Proof. Let $N_\pi(V_k) = \max\{\underline{N}(0), \underline{N}(1), \ldots, \underline{N}(k), \underline{N}(k+1)\}$. Then for $\forall\ N \geq N_\pi(V_k)$, $\pi(V_{k+1}, N) > \pi(V_k, N) > \cdots > \pi(c_{-1}, N)$. So $c^* \geq V_{k+1} > V_k$. □

Proof of Proposition 5

Proof. According to Equation 2.8 on page 347 in Feller (1968), for a given c, the conditional probability that an UP cascade occurs is 1 if $V = 1$, and the conditional probability that an UP cascade occurs is $\frac{q^k}{p^k}$ for V_{k-1} if $V = 0$.

When the project is bad, as $N \to \infty$, the probability that it is implemented without an UP cascade goes to 0. For any finite N and k, the probability that an UP cascade occurs is bounded from the top by $\frac{q^{k+1}}{p^{k+1}}$. Corollary 4 suggests that $\lim_{N\to\infty} c^*(N) = 1$, so $\lim_{N\to\infty} k \uparrow \infty$, and $\lim_{N\to\infty} \frac{q^{k+1}}{p^{k+1}} \downarrow 0$. Then the probability that a bad project is implemented converges to 0 as $N \to \infty$.

When the project is good, given Corollary 3, for each k, there exists a $N_\pi(V_k)$ such that for any $N \geq N_\pi(V_k)$, there exists a $c^* > V_k$ and $c^*P(c^*) > V_kP(V_k)$, where $P(c^*)$ is the unconditional probability that the project is implemented. Since $c^* < 1$, $P(c^*) > V_kP(V_k)$. As we just proved, the probability that a bad project is implemented converges to 0 as $N \to \infty$, so without loss of generality, $P(c^*|V = 1) > V_kP(V_k|V = 1)$ also holds given sufficient large N. For any given k, $\lim_{N\to\infty} P(V_k|V = 1) \uparrow 1$, so $P(c^*|V = 1) > V_k$ given sufficient large N. Then $\lim_{N\to\infty} P(c^*|V = 1) = 1$ given $\lim_{k\to\infty} V_k \uparrow 1$.

□

Proof of Proposition 8

Proof. Given N, c^*, and T_N^*, we show that an proposer's posterior belief on V is indeed increasing in the total amount raised.

We first note that before entering a cascade, the number of supporting agents equals the number of H signals. We use n to denote the the first n agents, and h the number of H signals up to that point. Then $k = 2h - n$.

Case 1: $N + k_c$ is even (which implies $k_c = 2T_N^* - N$).

Before or right at reaching the AoN threshold, $h \leq T_N^*$. We get k is at most $2T_N^* - N = k_c$, there is no cascade yet. k is increasing in h and $k = k_c$ when $h = T_N^*$. The posterior of V according to equation (7) is thus increasing in the number of supporters. This means if a project is not financed or barely reaches the AoN threshold, the proposer learns most substantially from the fundraising outcome about the true type of V.

After reaching the AoN threshold, $k > 2(T_N^* + 1) - N = k_c + 2$, a cascade must have started at the last agent or earlier. Since no information is accumulated during cascade, $k = k_c + 1$. This implies that $\mathbb{E}[V] = \mathbb{E}[V|k = k_c + 1]$ is flat for all $h > T_N^*$. Therefore, for projects that exceed the AoN threshold by a large margin, the proposer would not significantly positively update the belief on V beyond $\mathbb{E}[V|k = k_c + 1]$.

Case 2: $N + k_c$ is odd (which implies $k_c + 1 = 2T_N^* - N$).

Before or right at reaching the AoN threshold, $h < T_N^*$. We get k is at most $2(T_N^* - 1) - N = k_c - 1$, there is no cascade yet. k is increasing in h and $k = k_c - 1$ when $h = T_N^* - 1$. The posterior of V according to equation (7) is thus increasing in the number of supporters. This means if a project is not financed, the proposer learns most substantially from the fundraising outcome about the true type of V.

After reaching the AoN threshold, $k \geq 2(T_N^* + 1) - N = k_c + 1$, a cascade must have started at the last agent or earlier. Since no information is accumulated during cascade, $k = k_c + 1$. This implies that $\mathbb{E}[V] = \mathbb{E}[V|k = k_c + 1]$ is weakly increasing for $h \geq T_N^*$.

Taking all these into consideration, we conclude that the posterior of V is weakly increasing in total amount of supports observed (not necessarily received by the proposer). The sensitivity of the posterior belief on the total support is greater when the fundraising actually fails. $\square$

Proof of Proposition 9

Proof. First, suppose agent i observes H information, she has no incentive to deviate. If she chooses rejection or waiting, then all follow agents misinterpret her action and update their beliefs as if i observed L. This results in failures for some project that should be financed if i correctly reveals her information.

If agent i observes L, as we discussed in the baseline model, if there is an UP cascade she chooses to invest. When there is no UP cascade yet, she has no incentive to invest, and waiting is a weakly dominating strategy since she can always reject latter. Thus her first action of waiting still reveals her information. $\square$

Proof of Proposition 10

Proof. When agents have options to wait, investors with L signals invest if the project would be implemented. From the proof for Proposition 4, we have

$$\lim_{N\uparrow\infty}(V_k-\nu)\sum_{i=1}^{N-1}\varphi_{k+1,i+1}\geq \lim_{N\uparrow\infty}V_{k-1}\sum_{i=1}^{N}\varphi_{k,i}.$$

When N goes to infinity, since the probability that the project is implemented (reaching AoN threshold) without an Up-cascade goes to 0, the optimal choice of k goes to infinity. That is to say, c goes to 1.

Because when $V=1$, $Pr(X=H|V=1)=p>q$, it is known that (Feller (1968), page 347 equation 2.8)

$$Pr(C_o)=0.$$

UP cascade always takes place. So all good project would be implemented almost surely when N goes to infinity. □

Proof of Lemma 5

Proof. Any equilibrium involves a sub-game equilibrium given the proposer's decision on c and T. We only need to show that any sub-game equilibrium is either the informative one characterized in proposition 1 or one involves a group of free-riders whose actions before a cascade are ignored in equilibrium.

For any agent observing a H signal, it is her dominating strategy to contribute when there is a positive probability to reach the AoN threshold (and her action would be irrelevant if the project would not be implemented for sure). Given the tie-breaking assumption, in any sub-game equilibrium, agents with H signals always invest when there is a positive probability to reach the AoN threshold. For agent observing a L signal, if in the equilibrium given the history of actions the agent's action may depend on her private information, then she always reject when she observes L as discussed in the proof for proposition 1. If in the equilibrium given the history of actions the agent's action is independent of her private information, then her decision must be H. □

Proof for Proposition 11

Proof.

Definition 2. *For every proposal design (c,T), define $\pi(c,T)=\sup\pi^E(c,T)$ as the highest expected payoff the proposer gets among all possible subgame equilibria following this design. A proposal design (c,T) is called "optimistically optimal" if (c,T) maximizes $\pi(c,T)$.*

Lemma 9. *Suppose $\{(\hat{c}_N, T_N)\}_{N=1}^{\infty}$ is a sequence of optimistically optimal proposal design for positive integer sequence indexed by N. Then,*

$$\lim_{N\to\infty} \frac{\pi(\hat{c}_N, T_N)}{N} \to \frac{1}{2}$$

Proof. For a given N, design (c, T), and subgame equilibrium E, suppose the proposer receives $\pi_1^E(c, T)$ conditional on $V = 1$ and he receives $\pi_0^E(c, T)$ conditional on $V = 0$. The investors overall get $\frac{\pi_1^E(c,T)}{c}$ conditional on $V = 1$ and 0 conditional on $V = 0$. Because they never pay less than their expected value, $\pi_1^E(c, T)$ and $\pi_0^E(c, T)$ satisfy:

$$\frac{1}{2}(\frac{1}{c} - 1)\pi_1^E(c, T) - \frac{1}{2}\pi_0^E(c, T) \geq 0$$

As a result, we can find the following relation for the proposer's expected payoff

$$\pi^E(c, T) = \frac{1}{2}\pi_1^E(c, T) + \frac{1}{2}\pi_0^E(c, T) \leq \frac{1}{2m}\pi_1^E(c, T) < \frac{1}{2m}mN = \frac{N}{2} \Rightarrow \frac{\pi^E(c, T)}{N} < \frac{1}{2}$$
$$\Rightarrow \frac{\pi(\hat{c}_N, T_N)}{N} \leq \frac{1}{2} \qquad \forall N \in \mathbb{Z}^+$$

We next show that for every positive integer k, the proposer's expected payoff from proposal design $(V_k, \lceil\sqrt{N}\rceil)$ for N agents has the following limiting property:

$$\liminf_{N\to\infty} \frac{\pi(V_k, \lceil\sqrt{N}\rceil)}{N} \geq \frac{1}{2}V_k. \tag{26}$$

To see this, note that conditional on $V = 1$, all agents independently receive a high signal with probability p. Therefore, the law of large numbers imply $P(|\frac{H_{\lceil\sqrt{N}\rceil}(\mathcal{S})}{\lceil\sqrt{N}\rceil} - p| > \varepsilon | V = 1) \to 0$, where $H_n(\mathcal{S})$ is the number of high signals among the first n signals in $\mathcal{S}$. As such, for every $\varepsilon \in (0, p - \frac{1}{2})$ and positive integer k', there exists positive integer $N_{k,k'}$ such that:

$$\begin{aligned}
&P(|\frac{H_{\lceil\sqrt{N}\rceil}(\mathcal{S})}{\lceil\sqrt{N}\rceil} - p| > \varepsilon | V = 1) < \frac{1}{k'} \qquad \forall N \geq N_{k,k'} \\
\Rightarrow &P(\frac{H_{\lceil\sqrt{N}\rceil}(\mathcal{S})}{\lceil\sqrt{N}\rceil} - p < -\varepsilon | V = 1) < \frac{1}{k'} \\
\Rightarrow &P(2H_{\lceil\sqrt{N}\rceil}(\mathcal{S}) - \sqrt{N} - k - 1 < (-2\varepsilon + 2p - 1)\sqrt{N} - k - 1 | V = 1) < \frac{1}{k'}
\end{aligned} \tag{27}$$

Consequently, if we choose $N_{k,k'}$ big enough such that $(-2\varepsilon + 2p - 1)\sqrt{N_{k,k'}} - k - 1 > 0$, then the proposer's expected payoff from choosing $(V_k, \lceil\sqrt{N}\rceil)$ for $N > N_{k,k'}$ is at least:

$$\frac{\pi(V_k, \lceil\sqrt{N}\rceil)}{N} > \frac{1}{2}V_k\frac{N - \lceil\sqrt{N}\rceil}{N}P(2H_{\lceil\sqrt{N}\rceil}(\mathcal{S}) - \lceil\sqrt{N}\rceil \geq k + 1) > \frac{1}{2}\frac{V_k(N - \lceil\sqrt{N}\rceil)}{N}(1 - \frac{1}{k'})$$

In the second inequality, we used the fact that if $2H_n(\mathcal{S}) - n \geq k+1$, then an UP cascade starts at least by the nth agent. Taking N to infinity we get (26).

What (26) implies is that for every k, there exists a sufficiently large number N_k for which $\frac{\pi(\hat{c}_N, T_N)}{N} \geq \frac{\pi(V_k, \sqrt{N})}{N} \geq \frac{1}{2}V_k$, where $\pi(V_k, \sqrt{N})$ has unique subgame equilibrium by Corollary 1, and we know $\lim_{k\to\infty} V_k = 1$ from Corollary 4 and 5. Consequently, as N goes to infinity, $\frac{\pi(\hat{c}_N, T_N)}{N} \to \frac{1}{2}$. Then we complete the proof. $\square$

Definition 3. *For every proposal design (c,T), define $\tilde{\pi}(c,T) = \inf \pi^E(c,T)$ as the lowest expected payoff the proposer gets among all possible subgame equilibria following this design. A proposal design (c,T) is called "pessimistically optimal" if (c,T) maximizes $\tilde{\pi}(c,T)$.*

It should be obvious that $\tilde{\pi}(c,T)$ is bounded below by the optimal (c,T) where $c \in \{V_k, k = 0, 1, 2, \cdots\}$. To see this, suppose in a subgame equilibrium that the proposer's payoff is lower, then the proposer would profitably deviate to choosing an $c \in \{V_k, k = 0, 1, 2, \cdots\}$, for which we know has a unique subgame equilibrium.

As such, as $N \to \infty$, a sequence of pessimistically optimal proposal design for positive integer sequence indexed by N would also satisfy

$$\lim_{N\to\infty} \frac{\pi(\hat{c}_N, T_N)}{N} \to \frac{1}{2},$$

because that is the limit of when price is restricted to $c \in \{V_k, k = 0, 1, 2, \cdots\}$.

Given that no matter which equilibrium we have, the profit is in between the optimistically optimal design and the pessimistically optimal design, both of which converge to the same value, a sequence of proposer's endogenous design must also result in the per agent profit converging to $\frac{1}{2}$.

The next lemma shows that as N goes to infinity, the number of informers exceed any positive number with probability one.

Lemma 10. *For a given N, the corresponding proposal design (c,T), subgame equilibrium E, and a sequence of signals $\mathcal{S}$, define $nb^E(\mathcal{S};(c,T))$ as the number of informers given the sequence of signals. Moreover, suppose $\{\hat{c}_N, T_N\}_{N=1}^{\infty}$ is a sequence of endogenous designs, where the proposer's expected payoff is maximized for equilibrium E_N. Then for every positive integer l, $P(nb^{E_N}(\mathcal{S};(\hat{c}_N, T_N)) < l) \to 0$ as N goes to infinity.*

Proof. Consider the contrary and suppose for some $\varepsilon > 0$, $P(nb^{E_N}(\mathcal{S};(\hat{c}_N, T_N)) < l) > \varepsilon$ for infinite values of N. For such an N, we have:

$$\frac{\pi^{E_N}(\hat{c}_N, T_N)}{N} < \frac{1}{2N\hat{c}_N}\pi_1^{E_N}(\hat{c}_N, T_N) < \frac{1}{2N\hat{c}_N}(1 - \varepsilon(1-p)^l)N\hat{c}_N = \frac{1 - \varepsilon(1-p)^l}{2}$$

The second inequality follows from the fact that the proposal is not accepted when all the informers receive a low signal. But in Lemma 9, we showed that $\frac{\pi^{E_N}(\hat{c}_N, T_N)}{N}$ goes to $\frac{1}{2}$ in this sequence of numbers, which is a contradiction.

□

The above Lemma shows that as $N \to \infty$, there is always an arbitrarily high number of informers, which leads to the proposition. □

Proof for Proposition 12

Proof. Consider the contrary that there exists positive integer n such that $\hat{c}_N \leq c_n$ for all N. We want to show

$$\limsup_{N\to\infty} \frac{\pi(\hat{c}_N, T_N)}{N} \leq \frac{c_n}{2c_{n+1}} < \frac{1}{2} \tag{28}$$

which would contradict Lemma 9.

To see this, first note that if $\{T_N\}_{N=1}^{\infty}$ are bounded by some $\mathcal{T}$, then

$$\frac{\pi(\hat{c}_N, T_N)}{N} < \frac{1}{2N\hat{c}_N}\pi_1(\hat{c}_N, T_N) < \frac{1}{2N\hat{c}_N}(1-(1-p)^{\mathcal{T}})N\hat{c}_N = \frac{1-(1-p)^{\mathcal{T}}}{2}$$

, where the second inequality follows from the fact that the project would not be supported if all agents receive a low signal, even if $V = 1$. This contradicts Lemma 9, thus $\{T_N\}_{N=1}^{\infty}$ cannot be bounded. Next, it is easy to show that as T_N goes to infinity, the probability of accepting the proposal without reaching an UP cascade converges to zero. To see this, we define event $\mathbf{I}_A$ to be reaching AoN without an UP cascade when the endogenous design are $(\hat{c}_N, T_N)$, and event $\mathbf{I}_B$ to be that the number of informers is bigger than l for an arbitrary positive integer l. From Lemma 10, we know $Prob(\mathbf{I}_B) \to 1$ as $N \to \infty$.

$$\begin{aligned}
Prob(\mathbf{I}_A) \underbrace{\longrightarrow}_{N\to\infty} Prob(\mathbf{I}_A|\mathbf{I}_B) &\leq \mathbb{E}\left[\frac{1}{2}\binom{2I_N - k_N}{I_N}[p^{I_N}(1-p)^{I_N-k_N} + (1-p)^{I_N}p^{I_N-k_N}]|\mathbf{I}_B\right] \\
&< \mathbb{E}\left[\frac{1}{2}2^{2I_N-k_N}[p^{I_N}(1-p)^{I_N-k_N} + (1-p)^{I_N}p^{I_N-k_N}]|\mathbf{I}_B\right] \\
&< \mathbb{E}\left[\frac{1}{2}2^{2I_N-n}[2p^{I_N-n}(1-p)^{I_N-n}]|\mathbf{I}_B\right] = \mathbb{E}\left[(2p(1-p))^{2I_N-n}|\mathbf{I}_B\right] \\
&\underbrace{\longrightarrow}_{N\to\infty} 0
\end{aligned}$$

, where $\{k_N\}_{N=1}^{\infty}$ are such that $c_N \in (\mathbb{E}[V|K = k_N - 1], \mathbb{E}[V|K = k_N]]$, and by assumption are bounded above by n. the last limit follows from the fact that conditional on $\mathbf{I}_B$, the stochastic variable I_N on every path goes to infinity when N goes to infinity

As a result, when N goes to infinity, an UP cascade is reached with probability one conditional on the proposal getting enough support. Moreover, note that that once an UP cascade is reached for price $c_N = V_{k_N}$, the posterior belief about the project's success is V_{k_N+1}. Therefore, the investors' expected gross return conditional on reaching an UP cascade is $\frac{V_{k_N+1}}{V_{k_N}}$. In other words, the proposer can enjoy at most fraction $\frac{V_{k_N}}{V_{k_N+1}} \leq \frac{c_n}{c_{n+1}}$ of the surplus for large enough values of N. We have (28) because

$$\limsup_N \frac{\pi(\hat{c}_N, T_N)}{N} \leq \frac{c_n}{2c_{n+1}} < \frac{1}{2}$$

This completes the proof. $\square$